Carleman Estimates for Parabolic Operators with Discontinuous and Anisotropic Diffusion Coefficients, an Elementary Approach*

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Abstract

By using some deep tools from microlocal analysis, J. Le Rousseau and L. Robbiano (Invent. Math., 183 (2011), 245–336) established several Carleman estimates for parabolic operators with isotropic diffusion coefficients which have jumps at interfaces. In this paper, we revisit the same problem but for the general case of anisotropic diffusion coefficients. Our main tools are a pointwise estimate for parabolic operators and a suitable chosen weight function.

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Key Words. Carleman estimate, parabolic operator, diffusion coefficient, pointwise estimate, weight function.

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1 Introduction

Let $\Omega \in \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with the boundary $\Gamma \in C^2$, and $T > 0$. Let S be a C^2 hypersurface in Ω such that $\overline{S} \cap \Gamma = \emptyset$ and $\Omega \setminus S$ is composed of two connected domains Ω_1 and Ω_2. Denote by $\nu(x)(= (\nu_1(x), \cdots, \nu_n(x)))$ (*resp.* $\tilde{\nu}(x)$) the unit outward normal vector of Ω_1 (*resp.* Ω_2) at $x \in \Gamma_1$ (*resp.* $x \in \Gamma_2$) (Hence, $\nu(x) = -\tilde{\nu}(x)$ for any $x \in S$). For each $\delta > 0$ and $G \subset \mathbb{R}^n$, we define

$$O_\delta(G) = \{x \in \mathbb{R}^n \mid \mathrm{dist}\,(x, G) < \delta\}.$$

In what follows, we assume that

$$(a^{ij})_{1\le i,j\le n} \in C^2([0,T] \times \overline{\Omega_1}; \mathbb{R}^{n\times n}), \quad (\tilde{a}^{ij})_{1\le i,j\le n} \in C^2([0,T] \times \overline{\Omega_2}; \mathbb{R}^{n\times n}), \tag{1.1}$$

are fixed, satisfying

$$\begin{aligned} &a^{ij}(t,x) = a^{ji}(t,x), \qquad \forall\,(t,x) \in [0,T] \times \overline{\Omega_1},\ i,j = 1,2,\cdots,n,\\ &\tilde{a}^{ij}(t,x) = \tilde{a}^{ji}(t,x), \qquad \forall\,(t,x) \in [0,T] \times \overline{\Omega_2},\ i,j = 1,2,\cdots,n, \end{aligned} \tag{1.2}$$

and for some constant $s_0 > 0$,

$$\begin{aligned} &\sum_{i,j=1}^n a^{ij}(t,x)\xi^i\xi^j \ge s_0|\xi|^2, \qquad \forall\,(t,x,\xi) \in [0,T] \times \overline{\Omega_1} \times \mathbb{R}^n,\\ &\sum_{i,j=1}^n \tilde{a}^{ij}(t,x)\xi^i\xi^j \ge s_0|\xi|^2, \qquad \forall\,(t,x,\xi) \in [0,T] \times \overline{\Omega_2} \times \mathbb{R}^n, \end{aligned} \tag{1.3}$$

where $\xi = (\xi^1, \cdots, \xi^n)$.

We consider the following transmission problem for a parabolic equation:

$$\begin{cases} y_{1,t} + \displaystyle\sum_{i,j=1}^n (a^{ij} y_{1,x_i})_{x_j} = f_1 & \text{in } (0,T) \times \Omega_1,\\ y_{2,t} + \displaystyle\sum_{i,j=1}^n (\tilde{a}^{ij} y_{2,x_i})_{x_j} = f_2 & \text{in } (0,T) \times \Omega_2,\\ y_1 = 0 & \text{on } (0,T) \times (\Gamma_1 \setminus S),\\ y_2 = 0 & \text{on } (0,T) \times (\Gamma_2 \setminus S),\\ y_1 = y_2 + \beta_1 & \text{on } (0,T) \times S,\\ \displaystyle\sum_{i,j=1}^n a^{ij} y_{1,x_i}\nu_j = \sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i}\nu_j + \beta_2 & \text{on } (0,T) \times S,\\ y_1(T) = y_1^0 & \text{in } \Omega_1,\\ y_2(T) = y_2^0 & \text{in } \Omega_2, \end{cases} \tag{1.4}$$

where

$$\begin{cases} f_1 \in L^2((0,T)\times\Omega_1), \quad f_2 \in L^2((0,T)\times\Omega_2), \\ \beta_1 \in L^2(0,T;H^{\frac{3}{2}}(S)) \cap H^1(0,T;H^{\frac{1}{2}}(S)), \quad \beta_2 \in H^1(0,T;L^2(S)), \\ y_1^0 \in L^2(\Omega_1), \quad y_2^0 \in L^2(\Omega_2). \end{cases} \tag{1.5}$$

The equation (1.4) is well-posed in the class:

$$\Big\{(y_1,y_2) \,\Big|\, y_i \in C([0,T];L^2(\Omega_i)) \cap C((0,T];H^1(\Omega_i)), y_i|_{(0,T)\times(\Gamma_i\setminus S)} = 0, i=1,2\Big\}.$$

In what follows, we assume that ω is an open subset of Ω such that

$$\omega \cap \Omega_i \neq \emptyset, \quad i=1,2. \tag{1.6}$$

The main purpose of this paper is to derive the following type of *a priori* estimates, i.e., Carleman estimates for solutions to (1.4):

$$\begin{aligned} &\sum_{i=1}^{2}\int_0^T\int_{\Omega_i}(W_{i1}y_i^2 + W_{i2}|\nabla y_i|^2)dxdt \\ &\le C\Big[\sum_{i=1}^{2}\int_0^T\Big(\int_{\omega\cap\Omega_i}W_{i3}y_i^2 + \int_{\Omega_i}W_{i4}f_i^2\Big)dxdt \\ &\quad + |W_5\beta_1|^2_{L^2(0,T;H^{\frac{3}{2}}(S))\cap H^1(0,T;H^{\frac{1}{2}}(S))} + |W_6\beta_2|^2_{H^1(0,T;L^2(S))}\Big], \end{aligned} \tag{1.7}$$

for some suitably chosen (parameterized) weight functions W_{i1}, W_{i2}, W_{i3}, W_{i4} $(i=1,2)$, W_5 and W_6 (which are positive almost everywhere). Here and henceforth, C denotes a generic positive constant which may vary from one line to another.

Carleman estimate is a basic tool to solve many problems in partial differential equations, say uniqueness in Cauchy problems ([5, 15]), inverse problems ([6, 8]), control problems ([4, 13, 14]), and so on. Especially, there are many works addressing Carleman estimates for parabolic equations with smooth diffusion coefficients (e.g., [2, 4, 12]). Nevertheless, very little are known for the Carleman estimate for parabolic equations with non-smooth diffusion coefficients. In this respect, as far as we know, [1] is the first paper establishing global Carleman estimates for the equation (1.4) when

$$(a^{ij})_{1\le i,j\le n} = aI_n, \quad (\tilde{a}^{ij})_{1\le i,j\le n} = \tilde{a}I_n \tag{1.8}$$

for some time-independent functions $a \in C^2(\overline{\Omega_1})$ and $\tilde{a} \in C^2(\overline{\Omega_2})$ (I_n stands for the $n\times n$ identity matrix), and the following monotonicity condition holds:

$$a|_S \ge \tilde{a}|_S. \tag{1.9}$$

By means of the microlocal techniques, the recent work [7] obtained several interesting local and global Carleman estimates for the equation (1.4) when $S \in C^\infty$ and the assumption (1.8) holds for some time-dependent functions $a \in C^\infty([0,T]\times\overline{\Omega_1})$ and $\tilde{a} \in C^\infty([0,T]\times\overline{\Omega_2})$ without the monotonicity condition as (1.9).

In [7], an open problem was posed to derive Carleman estimates for the equation (1.4) in the presence of jumps of the diffusion (coefficient) matrix at the interface S. The present work aims to give an affirmative answer to this problem under considerably weak regularities on the involved data, say a^{ij}, $\tilde{a}^{ij}$ and S. For this purpose, we need to derive a careful pointwise estimate for parabolic operators, and to construct a suitable weight function that will be used in the Carleman estimates. It turns out that our approach is rather elementary.

The rest of this paper is organized as follows. In Section 2, as a preliminary, we establish a crucial pointwise estimate for a class of parabolic operators. As another key preliminary, we present a construction of the desired weight function in Section 3. Then, in Section 4, we show a global Carleman estimate for the equation (1.4) with time-independent diffusion coefficients. Finally, in Section 5, we derive local and global Carleman estimates for (1.4) with general diffusion coefficients.

2 A pointwise estimate for parabolic operators

Let $\widehat{\Omega} \subset \mathbb{R}^n$ be a bounded domain with the C^2-boundary. For any function $\psi \in C^2([0,T]\times\overline{\widehat{\Omega}})$, parameters $\lambda > 0$ and $\mu > 0$, and positive number $d > |\psi|_{L^\infty((0,T)\times\widehat{\Omega})}$, put

$$\varphi = \frac{e^{\mu\psi}}{t(T-t)}, \quad \alpha = \frac{e^{\mu\psi} - e^{\mu d}}{t(T-t)}, \quad \theta = e^{\lambda\alpha}. \tag{2.1}$$

In what follows, for a positive integer r, we denote by $O(\mu^r)$ a function of order μ^r for large μ (which is independent of λ and T); by $O_\mu(\lambda^r)$ a function of order λ^r for fixed μ and for large λ, which is independent of T, either.

For any $(b^{ij})_{1\le i,j\le n} \in C^1([0,T]\times\overline{\widehat{\Omega}};\mathbb{R}^{n\times n})$ satisfying

$$b^{ij}(t,x) = b^{ji}(t,x), \quad \forall\ (t,x) \in [0,T]\times\overline{\widehat{\Omega}},\ i,j = 1,2,\cdots,n, \tag{2.2}$$

we have the following pointwise estimate for the parabolic operator $\partial_t + \sum_{i,j=1}^n b^{ij}\partial_{x_i x_j}$.

Lemma 2.1 *Let $u \in C^2((0,T)\times\widehat{\Omega})$ and $v = \theta u$. Then, for any $\varepsilon > 0$, it holds that*

$$\begin{aligned}
&\theta^2\Big|u_t + \sum_{i,j=1}^n b^{ij}u_{x_i x_j}\Big|^2\\
&\ge \frac{\varepsilon}{8\lambda\varphi}|v_t|^2 + \operatorname{div} V + \frac{1}{2}M_t + \sum_{i,j=1}^n c^{ij}v_{x_i}v_{x_j} + Bv^2\\
&\quad -C\Big(\varepsilon\lambda\mu^2 + \frac{\lambda}{\varepsilon} + \lambda\mu + \lambda\mu + \mu^2\Big)\varphi|\nabla v|^2.
\end{aligned} \tag{2.3}$$

Here

$$M = \lambda^2\mu^2\varphi^2\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j}v^2 - \lambda\alpha_t v^2 - \sum_{i,j=1}^n b^{ij}v_{x_i}v_{x_j}, \tag{2.4}$$

$$\begin{cases} V &= (V^1, \cdots, V^n), \\ V^j &= \displaystyle\sum_{i=1}^n \Big[b^{ij} v_{x_i} v_t - \lambda\mu\varphi \Big(2b^{ij} v_{x_i} \sum_{k,\ell=1}^n b^{k\ell} \psi_{x_k} v_{x_\ell} - b^{ij}\psi_{x_i} \sum_{k,\ell=1}^n b^{k\ell} v_{x_k} v_{x_\ell} \Big) \\ & \quad \displaystyle - 2\lambda\mu^2\varphi b^{ij} v_{x_i} \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k}\psi_{x_\ell} v - \lambda^3\mu^3\varphi^3 b^{ij}\psi_{x_i} v^2 \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k}\psi_{x_\ell} \\ & \quad + \lambda^2\mu\varphi\alpha_t b^{ij}\psi_{x_i} v^2 \Big], \end{cases} \tag{2.5}$$

$$\begin{aligned} c^{ij} &= \lambda\mu\varphi \sum_{k,\ell=1}^n \Big(\mu b^{ij} b^{k\ell} \psi_{x_k}\psi_{x_\ell} - b^{ij}(b^{k\ell}\psi_{x_k})_{x_\ell} \ - b^{ij}_{x_k} b^{k\ell}\psi_{x_\ell} \Big) + b^{ij}_t \\ &= \lambda\mu^2\varphi b^{ij} \sum_{k,\ell=1}^n b^{k\ell}\psi_k\psi_\ell + \lambda\varphi O(\mu), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} B \geq\ & \lambda^3\mu^4\varphi^3 \Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} \Big)^2 + \lambda^3\mu^3\varphi^3 \Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k} \Big)_{x_\ell} \\ & - 2\lambda^2\mu^2\varphi(\varphi_t - \alpha_t) \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} - \frac{1}{2}\lambda^2\mu^2\varphi^2 \sum_{i,j=1}^n b^{ij}_t \psi_{x_i}\psi_{x_j} + \frac{1}{2}\lambda\alpha_{tt} \\ & -\lambda^2\mu\varphi\alpha_t \sum_{i,j=1}^n (b^{ij}\psi_{x_i})_{x_j} - 2\lambda^2\mu^2\varphi^2 \Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i x_j} \Big)^2 - 2\lambda^2\mu^4\varphi^2 \Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} \Big)^2 \\ & -C\lambda\mu^4\varphi - C\lambda^2\mu^4\varphi \\ =\ & \lambda^3\mu^4\varphi^3 \Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} \Big)^2 + \lambda^3\varphi^3 O(\mu^3) + \lambda^2\varphi^2 O(\mu^2) + O_\mu(\lambda^2). \end{aligned} \tag{2.7}$$

Proof: We borrow some idea from [3, 8, 9, 10]. It is an easy matter to see that

$$\begin{cases} \theta u_t = v_t - \lambda\alpha_t v, \quad \theta u_{x_i} = v_{x_i} - \lambda\mu\varphi\psi_{x_i} v, \\ \theta u_{x_i x_j} = v_{x_i x_j} - 2\lambda\mu\varphi\psi_{x_i} v_{x_j} + \lambda^2\mu^2\varphi^2\psi_{x_i}\psi_{x_j} v - \lambda\mu^2\varphi\psi_{x_i}\psi_{x_j} v - \lambda\mu\varphi\psi_{x_i x_j} v. \end{cases} \tag{2.8}$$

Write

$$\begin{cases} I_1 = \displaystyle\sum_{i,j=1}^n b^{ij} v_{x_i x_j} + \lambda^2\mu^2\varphi^2 \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v - \lambda\alpha_t v, \\ I_2 = v_t - 2\lambda\mu\varphi \displaystyle\sum_{i,j=1}^n b^{ij}\psi_{x_i} v_{x_j} - 2\lambda\mu^2\varphi \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v, \\ I_3 = \theta\Big(u_t + \displaystyle\sum_{i,j=1}^n b^{ij} u_{x_i x_j} \Big) - \lambda\mu^2\varphi \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v + \lambda\mu\varphi \sum_{i,j=1}^n b^{ij}\psi_{x_i x_j} v. \end{cases}$$

Then, we see $I_1 + I_2 = I_3$, which implies that

$$I_3^2 \geq I_2^2 + 2I_1 I_2. \tag{2.9}$$

By virtue of the Cauchy-Schwartz inequality, we have that

$$\begin{aligned} I_3^2 &\leq 2\theta^2\Big(u_t + \sum_{i,j=1}^n b^{ij} u_{x_i x_j}\Big)^2 + 4\lambda^2\mu^4\varphi^2\Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j}\Big)^2 v^2 \\ &\quad + 4\lambda^2\mu^2\varphi^2\Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i x_j}\Big)^2 v^2. \end{aligned} \tag{2.10}$$

Now,

$$\begin{aligned} I_1 I_2 &= \Big(\sum_{i,j=1}^n b^{ij} v_{x_i x_j} + \lambda^2\mu^2\varphi^2 \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v - \lambda\alpha_t v\Big) \\ &\quad \times\Big(v_t - 2\lambda\mu\varphi \sum_{i,j=1}^n b^{ij}\psi_{x_i} v_{x_j} - 2\lambda\mu^2\varphi \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v\Big) \\ &= \sum_{i,j=1}^n b^{ij} v_{x_i x_j} v_t - 2\lambda\mu\varphi \sum_{i,j=1}^n b^{ij} v_{x_i x_j} \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k} v_{x_\ell} \\ &\quad -2\lambda\mu^2\varphi \sum_{i,j=1}^n b^{ij} v_{x_i x_j} \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k}\psi_{x_\ell} v + \lambda^2\mu^2\varphi^2 \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v v_t \\ &\quad -2\lambda^3\mu^3\varphi^3 \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k} v_{x_\ell} - 2\lambda^3\mu^4\varphi^3\Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j}\Big)^2 v^2 \\ &\quad -\lambda\alpha_t v v_t + 2\lambda^2\mu\varphi\alpha_t \sum_{i,j=1}^n b^{ij}\psi_{x_i} v_{x_j} v + 2\lambda^2\mu^2\varphi\alpha_t \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v^2. \end{aligned} \tag{2.11}$$

Denote the terms in the right hand side of (2.11) by $J_i, i = 1, 2, \cdots, 9$, respectively. Then J_1 reads

$$\begin{aligned} J_1 &= \sum_{i,j=1}^n \big(b^{ij} v_{x_i} v_t\big)_{x_j} - \sum_{i,j=1}^n b^{ij}_{x_j} v_{x_i} v_t - \sum_{i,j=1}^n b^{ij} v_{x_i} v_{t x_j} \\ &= \sum_{i,j=1}^n \big(b^{ij} v_{x_i} v_t\big)_{x_j} - \sum_{i,j=1}^n b^{ij}_{x_j} v_{x_i} v_t - \frac{1}{2}\sum_{i,j=1}^n \big(b^{ij} v_{x_i} v_{x_j}\big)_t + \frac{1}{2}\sum_{i,j=1}^n b^{ij}_t v_{x_i} v_{x_j}. \end{aligned} \tag{2.12}$$

By (2.2), we see that

$$
\begin{aligned}
J_2 &= -2\lambda\mu\varphi\sum_{i,j=1}^{n} b^{ij}v_{x_ix_j}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}v_{x_\ell}\\
&= -\sum_{i,j=1}^{n}\Big(2\lambda\mu\varphi b^{ij}v_{x_i}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}v_{x_\ell}\Big)_{x_j} + 2\lambda\mu^2\varphi\Big(\sum_{i,j=1}^{n} b^{ij}\psi_{x_i}v_{x_j}\Big)^2\\
&\quad +2\lambda\mu\varphi\sum_{i,j=1}^{n}\Big[b^{ij}_{x_j}v_{x_i}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}v_{x_\ell} + b^{ij}v_{x_i}\sum_{k,\ell=1}^{n}(b^{k\ell}\psi_{x_k})_{x_j}v_{x_\ell}\\
&\quad +b^{ij}v_{x_i}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}v_{x_\ell x_j}\Big]\\
&= -\sum_{i,j=1}^{n}\Big(2\lambda\mu\varphi b^{ij}v_{x_i}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}v_{x_\ell} - \lambda\mu\varphi b^{ij}\psi_{x_i}\sum_{k,\ell=1}^{n} b^{k\ell}v_{x_k}v_{x_\ell}\Big)_{x_j}\\
&\quad +2\lambda\mu^2\varphi\Big(\sum_{i,j=1}^{n} b^{ij}\psi_{x_i}v_{x_j}\Big)^2 + 2\lambda\mu\varphi\sum_{i,j=1}^{n} b^{ij}_{x_j}v_{x_i}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}v_{x_\ell}\\
&\quad + 2\lambda\mu\varphi\sum_{i,j=1}^{n} b^{ij}v_{x_i}\sum_{k,\ell=1}^{n}(b^{k\ell}\psi_{x_k})_{x_j}v_{x_\ell} - \lambda\mu^2\varphi\sum_{i,j=1}^{n} b^{ij}v_{x_i}v_{x_j}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}\psi_{x_\ell}\\
&\quad - \lambda\mu\varphi\sum_{i,j=1}^{n} v_{x_i}v_{x_j}\sum_{k,\ell=1}^{n}(b^{k\ell}\psi_{x_k})_{x_\ell} - \lambda\mu\varphi\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} b^{ij}_{x_\ell}v_{x_i}v_{x_j}b^{k\ell}\psi_{x_k}.
\end{aligned}
\tag{2.13}
$$

Also,

$$
\begin{aligned}
J_3 &= -2\lambda\mu^2\varphi\sum_{i,j=1}^{n} b^{ij}v_{x_ix_j}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}\psi_{x_\ell}v\\
&= -\Big(\sum_{i,j=1}^{n} 2\lambda\mu^2\varphi b^{ij}v_{x_i}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}\psi_{x_\ell}v\Big)_{x_j} + 2\lambda\mu^2\varphi\sum_{i,j=1}^{n} b^{ij}v_{x_i}v_{x_j}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}\psi_{x_\ell}\\
&\quad +2\lambda\mu^2\varphi\sum_{i,j=1}^{n} b^{ij}_{x_j}v_{x_i}\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}\psi_{x_\ell}v + 2\lambda\mu^2\sum_{i,j=1}^{n} b^{ij}v_{x_i}v\Big(\varphi\sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k}\psi_{x_\ell}\Big)_{x_j}.
\end{aligned}
\tag{2.14}
$$

Further,

$$
\begin{aligned}
J_4 &= \lambda^2\mu^2\varphi^2\sum_{i,j=1}^{n} b^{ij}\psi_{x_i}\psi_{x_j}vv_t\\
&= \frac{1}{2}\Big(\lambda^2\mu^2\varphi^2\sum_{i,j=1}^{n} b^{ij}\psi_{x_i}\psi_{x_j}v^2\Big)_t - \lambda^2\mu^2\varphi\varphi_t\sum_{i,j=1}^{n} b^{ij}\psi_{x_i}\psi_{x_j}v^2\\
&\quad -\frac{1}{2}\lambda^2\mu^2\varphi^2\sum_{i,j=1}^{n} b^{ij}_t\psi_{x_i}\psi_{x_j}v^2 - \lambda^2\mu^2\varphi^2\sum_{i,j=1}^{n} b^{ij}\psi_{x_i}\psi_{x_jt}v^2.
\end{aligned}
\tag{2.15}
$$

The J_5 satisfies

$$
\begin{aligned}
J_5 &= -2\lambda^3\mu^3\varphi^3 \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k} v_{x_\ell} \\
&= -\sum_{i,j=1}^n \Big(\lambda^3\mu^3\varphi^3 b^{ij}\psi_{x_i}\psi_{x_j} \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k} v^2\Big)_{x_\ell} + 3\lambda^3\mu^4\varphi^3\Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j}\Big)^2 v^2 \\
&\quad +\lambda^3\mu^3\varphi^3\Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} \sum_{k,\ell=1}^n b^{k\ell}\psi_{x_k}\Big)_{x_\ell} v^2.
\end{aligned}
\tag{2.16}
$$

It is easy to see that

$$
J_6 = -2\lambda^3\mu^4\varphi^3\Big(\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j}\Big)^2 v^2, \tag{2.17}
$$

$$
J_7 = -\lambda\alpha_t v v_t = -\frac{1}{2}(\lambda\alpha_t v^2)_t + \frac{1}{2}\lambda\alpha_{tt} v^2, \tag{2.18}
$$

and

$$
J_9 = 2\lambda^2\mu^2\varphi\alpha_t \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v^2. \tag{2.19}
$$

Finally,

$$
\begin{aligned}
J_8 &= 2\lambda^2\mu\varphi\alpha_t \sum_{i,j=1}^n b^{ij}\psi_{x_i} v_{x_j} v \\
&= \Big(\lambda^2\mu\varphi\alpha_t \sum_{i,j=1}^n b^{ij}\psi_{x_i} v^2\Big)_{x_j} - (\lambda^2\mu\varphi)_{x_j}\alpha_t \sum_{i,j=1}^n b^{ij}\psi_{x_i} v^2 - \lambda^2\mu\varphi\alpha_{tx_j} \sum_{i,j=1}^n b^{ij}\psi_{x_i} v^2 \\
&\quad -\lambda^2\mu\varphi\alpha_t \sum_{i,j=1}^n (b^{ij}\psi_{x_i})_{x_j} v^2 \\
&= \Big(\lambda^2\mu\varphi\alpha_t \sum_{i,j=1}^n b^{ij}\psi_{x_i} v^2\Big)_{x_j} - \lambda^2\mu^2\varphi\alpha_t \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v^2 - \lambda^2\mu^2\varphi\varphi_t \sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v^2 \\
&\quad -\lambda^2\mu\varphi\alpha_t \sum_{i,j=1}^n (b^{ij}\psi_{x_i})_{x_j} v^2.
\end{aligned}
\tag{2.20}
$$

Noting that for any $a, b \in \mathbb{R}$, $|a+b|^2 \geq \frac{1}{2}a^2 - b^2$, we find that

$$
\frac{1}{2}I_2^2 \geq \frac{1}{4}|v_t|^2 - 4\lambda^2\mu^2\varphi^2\Big|\sum_{i,j=1}^n b^{ij}\psi_{x_i} v_{x_j}\Big|^2 - 4\lambda^2\mu^4\varphi^2\Big|\sum_{i,j=1}^n b^{ij}\psi_{x_i}\psi_{x_j} v\Big|^2. \tag{2.21}
$$

Consequently, we have for some $\varepsilon > 0$ (which is fixed but very small), that

$$
\begin{aligned}
\frac{\varepsilon}{2\lambda\varphi} I_2^2 &\geq \frac{\varepsilon}{4\lambda\varphi}|v_t|^2 - C\varepsilon\lambda\mu^2\varphi \sum_{i,j=1}^{n} |b^{ij}|^2_{C([0,T]\times\overline{\widetilde{\Omega}})} |\nabla\psi|^2 |\nabla v|^2 \\
&\quad - C\lambda\mu^4\varphi \sum_{i,j=1}^{n} |b^{ij}|^2_{C([0,T]\times\overline{\widetilde{\Omega}})} |\nabla\psi|^4 |v|^2 \\
&\geq \frac{\varepsilon}{4\lambda\varphi}|v_t|^2 - C\varepsilon\lambda\mu^2\varphi|\nabla v|^2 - C\lambda\mu^4\varphi|v|^2.
\end{aligned} \tag{2.22}
$$

Further,

$$
\begin{aligned}
\Big| \sum_{i,j=1}^{n} b^{ij}_{x_j} v_{x_i} v_t \Big| &\leq \frac{\varepsilon}{8\lambda\varphi}|v_t|^2 + \frac{C}{\varepsilon}\lambda\varphi \sum_{i,j=1}^{n} |\nabla b^{ij}|^2_{C([0,T]\times\overline{\widetilde{\Omega}})} |\nabla v|^2 \\
&\leq \frac{\varepsilon}{8\lambda\varphi}|v_t|^2 + \frac{C}{\varepsilon}\lambda\varphi|\nabla v|^2.
\end{aligned} \tag{2.23}
$$

Also,

$$
\begin{aligned}
&\Big| 2\lambda\mu^2\varphi \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \Big(b^{ij}_{x_j} b^{k\ell}\psi_{x_k}\psi_{x_\ell} + b^{ij}(b^{k\ell}\psi_{x_k}\psi_{x_\ell})_{x_j} + \mu b^{ij} b^{k\ell}\psi_{x_k}\psi_{x_\ell} \Big) v v_{x_i} \Big| \\
&\leq C\Big(\mu^2\varphi \sum_{i,j=1}^{n} |b^{ij}|_{C([0,T]\times\overline{\widetilde{\Omega}})} |\nabla v|^2 + \lambda^2\mu^4\varphi \sum_{i,j=1}^{n} |\nabla b^{ij}|_{C([0,T]\times\overline{\widetilde{\Omega}})} |v|^2 \Big) \\
&\leq C\mu^2\varphi|\nabla v|^2 + C\lambda^2\mu^4\varphi|v|^2,
\end{aligned} \tag{2.24}
$$

and

$$
\begin{aligned}
&\Big| 2\lambda\mu\varphi \sum_{i,j=1}^{n} b^{ij}_{x_j} v_{x_i} \sum_{k,\ell=1}^{n} b^{k\ell}\psi_{x_k} v_{x_\ell} + 2\lambda\mu\varphi \sum_{i,j=1}^{n} b^{ij} v_{x_i} \sum_{k,\ell=1}^{n} (b^{k\ell}\psi_{x_k})_{x_j} v_{x_\ell} \Big| \\
&\leq C\lambda\mu\varphi \sum_{i,j=1}^{n} |\nabla b^{ij}|_{C([0,T]\times\overline{\widetilde{\Omega}})} |\nabla v|^2 + C\lambda\mu\varphi \sum_{i,j=1}^{n} |b^{ij}|_{C([0,T]\times\overline{\widetilde{\Omega}})} |\nabla v|^2 \leq C\lambda\mu\varphi|\nabla v|^2.
\end{aligned} \tag{2.25}
$$

Combining (2.12)– (2.19), (2.9), (2.11), (2.21)–(2.25), we obtain the desired inequality (2.3). This completes the proof of Lemma 2.1. $\square$

3 Construction of the weight function

In this section, we present a construction of the weight function that will be used to establish the desired Carleman estimates. We have the following result.

Lemma 3.1 *Assume that* (1.1)–(1.3) *and* (1.6) *hold. Let $\omega_i \subset \omega\cap\Omega_i$ $(i = 1, 2)$ be nonempty open subsets such that $\overline{\omega_i} \subset \omega\cap\Omega_i$. Then, there exist a function $\tilde{\phi} \in C^2(\overline{\Omega_2})$, a positive integer L and functions $\phi^\ell \in C^2([t_\ell, t_{\ell+1}] \times \overline{\Omega_1})$ with $t_\ell = \frac{\ell T}{L}$, $\ell = 0, 1, \cdots, L$, such that*

(1) $\phi^\ell > 0$ *in* $[t_\ell, t_{\ell+1}] \times \Omega_1$ *and* $\tilde\phi > 0$ *in* Ω_2;

(2) $\phi^\ell = 0$ *on* $[t_\ell, t_{\ell+1}] \times \Gamma_1$ *and* $\tilde\phi = 0$ *on* Γ_2;

(3) $|\nabla \phi^\ell| > 0$ *in* $[t_\ell, t_{\ell+1}] \times \overline{\Omega_1 \setminus \omega_1}$ *and* $|\nabla\tilde\phi| > 0$ *in* $\overline{\Omega_2 \setminus \omega_2}$;

(4) $\displaystyle\sum_{i,j=1}^n a^{ij}\phi^\ell_{x_i}\phi^\ell_{x_j} = \sum_{i,j=1}^n \tilde a^{ij}\tilde\phi_{x_i}\tilde\phi_{x_j}$ *on* $[t_\ell, t_{\ell+1}] \times S$.

Proof: By [4, Lemma 1.1, p.4], there is a function $\tilde\phi \in C^2(\overline{\Omega_2})$ such that

$$\begin{cases} \tilde\phi > 0 & \text{in } \Omega_2, \\ \tilde\phi = 0 & \text{on } \Gamma_2, \\ |\nabla\tilde\phi| > 0 & \text{in } \overline{\Omega_2 \setminus \omega_2}. \end{cases} \tag{3.1}$$

Let us extend $\tilde a^{ij}$ $(1 \le i,j \le n)$ to be a C^2 function on $[0,T]\times(O_\varepsilon(S)\cup\Omega_2)$ and denote by $\tilde a^{ij}$ the extended function, where ε is a sufficiently small positive number such that $(\tilde a^{ij})_{1\le i,j\le n}$ is still a uniformly positive definite matrix on $[0,T]\times(O_\varepsilon(S)\cup\Omega_2)$. Further, we extend the above $\tilde\phi$ to be a C^2 function on $O_\varepsilon(S)\cup\Omega_2$ and denote by $\tilde\phi$ itself the extension. Since $\frac{\partial\tilde\phi}{\partial\nu} > 0$ on S (Recall that $\nu(x) = -\tilde\nu(x)$ for $x \in S$), there is an $\varepsilon_1 \in (0,\varepsilon]$ so that

$$\sum_{i,j=1}^n \tilde a^{ij}\tilde\phi_{x_i}\tilde\phi_{x_j} > 0, \qquad \text{for all } (t,x) \in [0,T]\times O_{\varepsilon_1}(S).$$

Let us choose a (time-independent) $\hat\xi \in C^2(\overline{\Omega_1})$ such that $\hat\xi > 0$ in Ω_1 and $\hat\xi = 0$ on Γ_1. It follows from $\frac{\partial\hat\xi}{\partial\nu} < 0$ on S that there is an $\varepsilon_2 \in (0,\varepsilon_1]$ such that

$$\sum_{i,j=1}^n a^{ij}\hat\xi_{x_i}\hat\xi_{x_j} > 0, \qquad \text{for all } (t,x) \in [0,T]\times O_{\varepsilon_2}(S).$$

Then, we see that

$$\varsigma \triangleq \Big(\frac{\sum_{i,j=1}^n \tilde a^{ij}\tilde\phi_{x_i}\tilde\phi_{x_j}}{\sum_{i,j=1}^n a^{ij}\hat\xi_{x_i}\hat\xi_{x_j}}\Big)^{\frac12} > 0 \quad \text{in } [0,T]\times(\overline{\Omega_1}\cap O_{\varepsilon_2}(S))$$

and $\varsigma \in C^2([0,T]\times(\overline{\Omega_1}\cap O_{\varepsilon_2}(S)))$. We extend ς to be a positive C^2 function on $[0,T]\times\overline{\Omega_1}$, and still denote by ς the extension.

Put $\xi = \varsigma\hat\xi$. Then, $\xi \in C^2([0,T]\times\overline{\Omega_1})$, $\xi > 0$ in $[0,T]\times\Omega_1$ and $\xi = 0$ on $[0,T]\times\Gamma_1$. Further,

$$\sum_{i,j=1}^n a^{ij}\xi_{x_i}\xi_{x_j} = \varsigma^2\sum_{i,j=1}^n a^{ij}\hat\xi_{x_i}\hat\xi_{x_j} = \sum_{i,j=1}^n \tilde a^{ij}\tilde\phi_{x_i}\tilde\phi_{x_j}, \quad \text{for each } (t,x)\in[0,T]\times S. \tag{3.2}$$

In what follows, we will construct the desired function ϕ^ℓ based on ξ. The method is very similar to that of [4, Lemma 1.1, p.4 and pp. 20–22]. However, we give the detail here for the sake of completeness.

First, by $\frac{\partial\xi}{\partial\nu}<0$ on $[0,T]\times S$, we may find a small $\delta_1>0$ such that $\overline{O_{\delta_1}(\Gamma_1)}\cap\overline{\omega_1}=\emptyset$ and

$$c_1 \stackrel{\triangle}{=} \min_{[t,x]\in[0,T]\times(\Omega_1\setminus\overline{O_{\delta_1/2}(\Gamma_1)})} \xi>0, \qquad c_2 \stackrel{\triangle}{=} \min_{[t,x]\in[0,T]\times(\Omega_1\cap O_{\delta_1}(\Gamma_1))} |\nabla\xi|>0. \tag{3.3}$$

Let $\chi\in C^\infty(\overline{\Omega_1};[0,1])$ be such that $\chi=1$ in $O_{\delta_1/2}(\Gamma_1)\cap\Omega_1$ and $\chi=0$ in $\Omega_1\setminus\overline{O_{\delta_1}(\Gamma_1)}$. By $\xi\in C^2([0,T]\times\overline{\Omega_1})\subset C([0,T];C^1(\overline{\Omega_1}))$, we may find a (large) positive integer L such that

$$|\xi(t,\cdot)-\xi(s,\cdot)|_{C^1(\overline{\Omega_1})}<\min\Big(\frac{c_1}{2},\frac{c_2}{2|\chi|_{C^1(\overline{\Omega_1})}}\Big), \qquad \forall\, t,s\in[t_\ell,t_{\ell+1}], \tag{3.4}$$

where $t_\ell=\frac{\ell T}{L}$, $\ell=0,1,\cdots,L$.

Put

$$\mathcal{M}=\big\{x\in\Omega_1\;\big|\;\nabla\xi(t,x)=0 \text{ for some } t\in[0,T]\big\}.$$

By (3.3), we may choose an $O(\mathcal{M})(\subset\Omega_1)$ to be a neighborhood of $\mathcal{M}$ such that $\overline{O(\mathcal{M})}\subset\Omega_1\setminus\overline{O_{\delta_1}(\Gamma_1)}$. By virtue of the density of Morse functions in $C^2(\overline{\Omega}_1)$ (see [11, page 37]), for the above t_ℓ, one can find a sequence of Morse function $\{\xi_k^\ell\}_{k=1}^\infty$, such that $\xi_k^\ell(\cdot)$ converges to $\xi(t_\ell,\cdot)$ strongly in $C^2(\overline{\Omega}_1)$. Let

$$\rho_k^\ell(t,x)=\xi_k^\ell(x)+\chi(x)[\xi(t,x)-\xi_k^\ell(x)]. \tag{3.5}$$

Obviously, for any $t\in[0,T]$,

$$\rho_k^\ell(t,x)=\begin{cases}\xi(t,x), & x\in O_{\delta_1/2}(\Gamma_1)\cap\Omega_1,\\ \xi_k^\ell(x), & x\in\Omega_1\setminus\overline{O_{\delta_1}(\Gamma_1)},\end{cases} \tag{3.6}$$

and $\rho_k^\ell(t_\ell,\cdot)$ converges to $\xi(t_\ell,\cdot)$ strongly in $C^2(\overline{\Omega_1})$. By (3.4), it follows that

$$\begin{aligned}\rho_k^\ell(t,x)&=\xi(t_\ell,x)+[1-\chi(x)][\xi_k^\ell(x)-\xi(t_\ell,x)]+\chi(x)[\xi(t,x)-\xi(t_\ell,x)]\\&\geq \xi(t_\ell,x)-|\xi_k^\ell(\cdot)-\xi(t_\ell,\cdot)|_{C(\overline{\Omega_1})}-|\xi(t,\cdot)-\xi(t_\ell,\cdot)|_{C^1(\overline{\Omega_1})}\\&\geq c_1-|\xi_k^\ell(\cdot)-\xi(t_\ell,\cdot)|_{C(\overline{\Omega_1})}-|\xi(t,\cdot)-\xi(t_\ell,\cdot)|_{C^1(\overline{\Omega_1})}\\&\geq \frac{c_1}{2}-|\xi_k(^\ell\cdot)-\xi(t_\ell,\cdot)|_{C(\overline{\Omega_1})}, \quad \forall\,[t,x]\in[t_\ell,t_{\ell+1}]\times(\Omega_1\setminus\overline{O_{\delta_1/2}(\Gamma_1)}).\end{aligned} \tag{3.7}$$

Hence, there is a sufficiently large $k_0>0$ such that for any $k\geq k_0$,

$$\begin{cases}\xi_k^\ell>0 & \text{in } \Omega_1\setminus\overline{O_{\delta_1/2}(\Gamma_1)},\\ |\nabla\xi_k^\ell|>0 & \text{in } \Omega_1\cap O_{\delta_1}(\Gamma_1),\\ \rho_k^\ell>0 & \text{in } [t_\ell,t_{\ell+1}]\times\Omega_1.\end{cases} \tag{3.8}$$

By (3.4) again, we see that

$$
\begin{aligned}
|\nabla\rho_k^\ell(t,x)| &= |\nabla\xi_k^\ell(x)+\nabla\chi[\xi(t,x)-\xi_k^\ell(x)]+\chi\nabla[\xi(t,x)-\xi_k^\ell(x)]| \\
&\geq |\nabla\xi_k^\ell(x)|-|\chi|_{C^1(\overline{\Omega_1})}|\xi(t,\cdot)-\xi_k^\ell(\cdot)|_{C^1(\overline{\Omega_1})} \\
&\geq |\nabla\xi(t_\ell,x)|-|\nabla\xi(t_\ell,x)-\nabla\xi_k^\ell(x)|-|\chi|_{C^1(\overline{\Omega_1})}|\xi(t_\ell,\cdot)-\xi_k^\ell(\cdot)|_{C^1(\overline{\Omega_1})} \\
&\quad -|\chi|_{C^1(\overline{\Omega_1})}|\xi(t_\ell,\cdot)-\xi(t,\cdot)|_{C^1(\overline{\Omega_1})} \\
&\geq |\nabla\xi(t_\ell,x)|-(1+|\chi|_{C^1(\overline{\Omega_1})})|\xi(t_\ell,\cdot)-\xi_k^\ell(\cdot)|_{C^1(\overline{\Omega_1})} \\
&\quad -|\chi|_{C^1(\overline{\Omega_1})}|\xi(t_\ell,\cdot)-\xi(t,\cdot)|_{C^1(\overline{\Omega_1})} \\
&\geq \frac{c_2}{2}-(1+|\chi|_{C^1(\overline{\Omega_1})})|\xi(t_\ell,\cdot)-\xi_k^\ell(\cdot)|_{C^1(\overline{\Omega_1})}, \\
&\qquad\qquad\qquad\qquad \forall\,[t,x]\in[t_\ell,t_{\ell+1}]\times(\Omega_1\cap O_{\delta_1}(\Gamma_1)).
\end{aligned}
\tag{3.9}
$$

Hence, there is a sufficiently large $k_1>k_0$ such that for any $k\geq k_1$, it holds

$$
|\nabla\rho_k^\ell|>0 \quad \text{in } [t_\ell,t_{\ell+1}]\times(\Omega_1\cap O_{\delta_1}(\Gamma_1)). \tag{3.10}
$$

Put $\rho^\ell=\rho_k^\ell$. Noting that ξ_k^ℓ is a Morse function, by (3.6) and (3.10), we see that $\rho^\ell(\cdot,\cdot)$ is a Morse function with respect to x in the cylinder $[t_\ell,t_{\ell+1}]\times\Omega_1$ and

$$
\{x\in\Omega_1 \mid |\nabla\rho^\ell(t,x)|=0\}\subset\Omega_1\setminus\overline{O_{\delta_1}(\Gamma_1)}, \quad \text{for all } t\in[t_\ell,t_{\ell+1}].
$$

Utilizing (3.6), (3.8) and (3.10), we find that

$$
\{x\in\Omega_1 \mid |\nabla\rho^\ell(t,x)|=0\}=\{x\in\Omega_1 \mid |\nabla\xi_k^\ell(x)|=0\}, \quad \text{for all } t\in[t_\ell,t_{\ell+1}]. \tag{3.11}
$$

Further, by (3.2), (3.6) and (3.8), it is easy to see that

$$
\begin{cases}
\rho^\ell>0 & \text{in } [t_\ell,t_{\ell+1}]\times\Omega_1, \\
\rho^\ell=0 & \text{on } [t_\ell,t_{\ell+1}]\times\Gamma_1, \\
\displaystyle\sum_{i,j=1}^n a^{ij}\rho^\ell_{x_i}\rho^\ell_{x_j}=\sum_{i,j=1}^n \tilde a^{ij}\tilde\phi_{x_i}\tilde\phi_{x_j} & \text{on } [t_\ell,t_{\ell+1}]\times S.
\end{cases}
\tag{3.12}
$$

Denote by $\mathcal{P}=\{x_1,x_2,\cdots,x_m\}$ the set of critical points of ρ^ℓ with respect to x in $[t_\ell,t_{\ell+1}]\times\Omega_1$ and by $\{y_1,y_2,\cdots,y_m\}$ a subset of ω_1. Note that, in view of (3.11), $\mathcal{P}$ is actually independent of t. Let $\gamma_1,\gamma_2,\cdots,\gamma_m$ be simple and C^∞-curves in Ω_1 such that

$$
\begin{cases}
\gamma_i(s)\in\Omega_1,\ \ \forall\, s\in[0,1], \gamma_i(s_1)\neq\gamma_i(s_2), \quad \forall\, s_1,s_2\in[0,1] \text{ and } s_1\neq s_2\ i=1,2,\cdots,m; \\
\gamma_i(1)=x_i, \gamma_i(0)=y_i \quad i=1,2,\cdots,m; \\
\gamma_i(s_1)\neq\gamma_j(s_2), \quad \forall\, i\neq j, s_1\neq s_2\in[0,1].
\end{cases}
\tag{3.13}
$$

Then there exist a sequence of functions $\{\eta_i\}_{i=1}^m \subset C^2(\Omega_1;\mathbb{R}^n)$ and $\{\varrho_i\}_{i=1}^m \subset C_0^\infty(\Omega_1 \setminus \overline{O_{\delta_1}(\Gamma_1)})$ such that

$$\begin{cases} \dfrac{d\gamma_i(s)}{ds} = \eta_i(\gamma_i(s)) \quad \text{ in } [0,1],\ i=1,2,\cdots,m; \\ \operatorname{supp}\varrho_i \cap \operatorname{supp}\varrho_j = \emptyset, \quad i \neq j; \\ \varrho_i(\gamma_i(s)) = 1 \quad \text{ in } [0,1],\ i=1,2,\cdots,m. \end{cases} \tag{3.14}$$

Let

$$V_i(x) = \varrho_i(x)\eta_i(x).$$

Let us consider the following ordinary differential equation:

$$\begin{cases} \dfrac{dx}{ds} = V_i(x), \quad s \geq 0, \\ x(0) = x_0. \end{cases} \tag{3.15}$$

This defines an operator $S_i(\cdot) : \Omega_1 \to \Omega_1$ as $S_i(s)(x_0) = x(s)$, where $x(s)$ solves (3.15).

It follows from (3.13)–(3.15) that for any $s \in [0,T]$,

$$S_j(s)(y_i) = y_i, \quad i,j = 1,2,\cdots,m,\ i \neq j.$$

Let

$$\phi^\ell(x) = \rho^\ell(g_m(x)), \quad g_m(x) = S_1(1) \circ S_2(1) \circ S_3(1) \circ \cdots \circ S_m(1)x. \tag{3.16}$$

By means of the second condition in (3.14), we conclude that

$$S_i(1)x = x, \quad \forall\, x \in O_{\delta_1}(\Gamma_1) \cap \Omega_1, \quad i = 1,2,\cdots,m. \tag{3.17}$$

Therefore,

$$\begin{cases} \phi^\ell = 0 & \text{on } [t_\ell, t_{\ell+1}] \times \Gamma_1; \\ \displaystyle\sum_{i,j=1}^n a^{ij}\phi^\ell_{x_i}\phi^\ell_{x_j} = \sum_{i,j=1}^n \tilde{a}^{ij}\tilde{\phi}_{x_i}\tilde{\phi}_{x_j} & \text{on } [t_\ell, t_{\ell+1}] \times S. \end{cases} \tag{3.18}$$

Denote by Φ the set of critical points of ϕ^ℓ. Since $g_m(\cdot) : \Omega_1 \to \Omega_1$ is a diffeomorfism, we have

$$\Phi = \big\{x \in \Omega_1 \,\big|\, g_m(x) \in \mathcal{P}\big\}. \tag{3.19}$$

By (3.14) and (3.17), we have

$$g_m(y_i) = g_m(\gamma_i(0)) = x_i \in \mathcal{P}, \quad i = 1,2,\cdots,m. \tag{3.20}$$

Finally, from (3.19) and (3.20), we find that $\Phi \subset \omega_1$. Then, we see

$$|\nabla\phi^\ell| > 0 \text{ in } [t_\ell, t_{\ell+1}] \times (\Omega_1 \setminus \omega_1),$$

which completes the proof. $\square$

4 The case of time-independent diffusion coefficients

In this section, we consider the special case that both $(a^{ij})_{1\le i,j\le n}$ and $(\tilde a^{ij})_{1\le i,j\le n}$ are time-independent. In this case, from the proof of Lemma 3.1, it is easy to see that there exist two (time-independent) functions $\phi \in C^2(\overline{\Omega_1})$ and $\tilde\phi \in C^2(\overline{\Omega_2})$ such that: (1) $\phi > 0$ in Ω_1 and $\tilde\phi > 0$ in Ω_2; (2) $\phi = 0$ on Γ_1 and $\tilde\phi = 0$ on Γ_2; (3) $|\nabla\phi| > 0$ in $\overline{\Omega_1 \setminus \omega_1}$ and $|\nabla\tilde\phi| > 0$ in $\overline{\Omega_2 \setminus \omega_2}$; and (4) $\sum\limits_{i,j=1}^n a^{ij}\phi_{x_i}\phi_{x_j} = \sum\limits_{i,j=1}^n \tilde a^{ij}\tilde\phi_{x_i}\tilde\phi_{x_j}$ on S.

For any parameters $\lambda > 0$ and $\mu > 0$, put

$$
\begin{aligned}
&\varphi = \frac{e^{\mu\phi}}{t(T-t)}, \quad \alpha = \frac{e^{\mu\phi} - e^{\mu(|\phi|_{L^\infty(\Omega_1)} + |\tilde\phi|_{L^\infty(\Omega_2)})}}{t(T-t)}, \quad \theta = e^{\lambda\alpha},\\
&\tilde\varphi = \frac{e^{\mu\tilde\phi}}{t(T-t)}, \quad \tilde\alpha = \frac{e^{\mu\tilde\phi} - e^{\mu(|\phi|_{L^\infty(\Omega_1)} + |\tilde\phi|_{L^\infty(\Omega_2)})}}{t(T-t)}, \quad \tilde\theta = e^{\lambda\tilde\alpha}.
\end{aligned}
\tag{4.1}
$$

We have the following global Carleman estimate for solutions to the equation (1.4).

Theorem 4.1 *Assume that the diffusion coefficients $(a^{ij})_{1\le i,j\le n}$ and $(\tilde a^{ij})_{1\le i,j\le n}$ are time-independent, and (1.1)–(1.3) and (1.6) hold. Then, there exists a $\mu_0 > 0$ such that for any $\mu \ge \mu_0$, one can find a $\lambda_0 = \lambda_0(\mu) > 0$ so that for any $\lambda \ge \lambda_0$, and each y_i^0 $(i = 1, 2)$, f_i and β_i satisfying (1.5), the corresponding solution to (1.4) satisfies*

$$
\begin{aligned}
&C\int_0^T\int_{\Omega_1}\theta^2 f_1^2 dxdt + C\lambda^3\mu^4\int_0^T\int_{\omega\cap\Omega_1}\theta^2\varphi^3 y_1^2 dxdt\\
&+C\int_0^T\int_{\Omega_2}\tilde\theta^2 f_2^2 dxdt + C\lambda^3\mu^4\int_0^T\int_{\omega\cap\Omega_2}\tilde\theta^2\tilde\varphi^3 y_2^2 dxdt\\
&+C\lambda\int_0^T\left[\left|\tilde\theta\tilde\varphi^{\frac12}\beta_{1,t}\right|^2_{H^{\frac12}(S)} + \lambda^2\left|\tilde\theta\tilde\varphi^{\frac12}|\tilde\alpha_t|\beta_1\right|^2_{H^{\frac32}(S)} + \lambda^4\mu^4\left|\tilde\theta\tilde\varphi^{\frac52}\beta_1\right|^2_{H^{\frac32}(S)}\right]dt\\
&+C\int_0^T\int_S\frac{\theta^2|\alpha_t|^2}{\mu\varphi}\beta_2^2 dSdt + \int_0^T\int_S\frac{\theta^2}{\lambda\mu\varphi}|\beta_{2,t}|^2 dSdt + C\lambda^3\mu^3\int_0^T\int_S\theta^2\varphi^3\beta_2^2 dSdt\\
\ge&\int_0^T\int_{\Omega_1}\frac{1}{\lambda\varphi}\theta^2|y_{1,t}|^2 dxdt + \int_0^T\int_{\Omega_1}\theta^2(\lambda\mu^2\varphi|\nabla y_1|^2 + \lambda^3\mu^4\varphi^3 y_1^2)dxdt\\
&+\int_0^T\int_{\Omega_2}\frac{1}{\lambda\tilde\varphi}\tilde\theta^2|y_{2,t}|^2 dxdt + \int_0^T\int_{\Omega_2}\tilde\theta^2(\lambda\mu^2\tilde\varphi|\nabla y_2|^2 + \lambda^3\mu^4\tilde\varphi^3 y_2^2)dxdt.
\end{aligned}
\tag{4.2}
$$

Proof. We divide the proof into several steps.

Step 1. Let $\widehat\Omega = \Omega_1$, $u = y_1$, $v = \theta y_1$, $(b^{ij})_{1\le i,j\le n} = (a^{ij})_{1\le i,j\le n}$, $\psi = \phi$ and $d = |\phi|_{L^\infty(\Omega_1)} + |\tilde\phi|_{L^\infty(\Omega_2)}$ in Lemma 2.1. By Lemma 3.1, there are a $\mu_1 > 0$ and a $\lambda_1 = \lambda_1(\mu_1) > 0$ so that for all $\mu \ge \mu_1$ and $\lambda \ge \lambda_1$, it holds that (Recall (2.6) and (2.7) for c^{ij} and B, respectively)

$$
\begin{cases}
c^{ij} \ge C\lambda\mu^2\varphi b^{ij} & \text{in } (0,T)\times(\Omega_1\setminus\omega_1),\\
B \ge C\lambda^3\mu^4\varphi^3 & \text{in } (0,T)\times(\Omega_1\setminus\omega_1).
\end{cases}
\tag{4.3}
$$

Thus, one can find a sufficiently small $\varepsilon > 0$, a $C_1 > 0$, a $\mu_2 = \mu_2(\varepsilon) \geq \mu_1$ and a $\lambda_2 = \lambda_2(\mu_1) \geq \lambda_2$, such that for all $\mu \geq \mu_2$ and $\lambda \geq \lambda_2$, it holds that

$$\sum_{i,j=1}^{n} c^{ij} v_{x_i} v_{x_j} - C\Big(\varepsilon\lambda\mu^2 + \frac{\lambda}{\varepsilon} + \lambda\mu + \lambda\mu + \mu^2\Big)\varphi|\nabla v|^2 \geq C_1\lambda\mu^2\varphi|\nabla v|^2, \quad \text{in } (0,T)\times(\Omega_1\backslash\omega_1). \tag{4.4}$$

From (4.3) and (4.4), one can find a positive constant $C_2 = C_2(\varepsilon)$, such that for all $\mu \geq \mu_2$ and $\lambda \geq \lambda_2$, it holds that

$$\begin{aligned}
&\sum_{i,j=1}^{n} c^{ij} v_{x_i} v_{x_j} + Bv^2 - C\Big(\varepsilon\lambda\mu^2 + \frac{\lambda}{\varepsilon} + \lambda\mu + \lambda\mu + \mu^2\Big)\varphi|\nabla v|^2 \\
&\geq C_2(\lambda\mu^2\varphi|\nabla v|^2 + \lambda^3\mu^4\varphi^3 v^2) \ \text{ in } (0,T) \times (\Omega_1 \setminus \omega_1).
\end{aligned} \tag{4.5}$$

Then, by integrating the inequality (2.3) in $(0,T) \times \Omega_1$ and noting (4.5), from (2.4)–(2.5) and the choice of θ, we find that

$$\begin{aligned}
&C\int_0^T\int_{\Omega_1} \theta^2 f_1^2 dxdt + C\int_0^T\int_{\omega_1} (\lambda\mu^2\varphi|\nabla v|^2 + \lambda^3\mu^4\varphi^3 v^2)dxdt \\
\geq &\int_0^T\int_{\Omega_1} \frac{\varepsilon}{8\lambda\varphi}|v_t|^2 dxdt + \int_0^T\int_{\Omega_1} \operatorname{div} V dxdt \\
&+C_2\int_0^T\int_{\Omega_1} (\lambda\mu^2\varphi|\nabla v|^2 + \lambda^3\mu^4\varphi^3 v^2)dxdt,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
&\int_0^T\int_{\Omega_1} \operatorname{div} V dxdt \\
=&\int_0^T\int_{\Gamma_1}\sum_{i,j=1}^{n}\Big[a^{ij}v_{x_i}v_t - \lambda\mu\varphi\Big(2a^{ij}v_{x_i}\sum_{k,\ell=1}^{n}a^{k\ell}\phi_{x_k}v_{x_\ell} - a^{ij}\phi_{x_i}\sum_{k,\ell=1}^{n}a^{k\ell}v_{x_k}v_{x_\ell}\Big) \\
&- 2\lambda\mu^2\varphi a^{ij}v_{x_i}\sum_{k,\ell=1}^{n}a^{k\ell}\phi_{x_k}\phi_{x_\ell}v - \lambda^3\mu^3\varphi^3 a^{ij}\phi_{x_i}v^2\sum_{k,\ell=1}^{n}a^{k\ell}\phi_{x_k}\phi_{x_\ell} \\
&+\lambda^2\mu\varphi\alpha_t a^{ij}\phi_{x_i}v^2\Big]\nu_j d\Gamma_1 dt.
\end{aligned} \tag{4.7}$$

Now we analyze the terms in the right hand side of (4.7) one by one. First,

$$\begin{aligned}
&\int_0^T\int_{\Gamma_1}\sum_{i,j=1}^{n} a^{ij}v_{x_i}v_t\nu_j d\Gamma_1 dt \\
=&\int_0^T\int_S \theta^2\sum_{i,j=1}^{n} a^{ij}(\lambda\mu\varphi\phi_{x_i}y_1 + y_{1,x_i})(\lambda\alpha_t y_1 + y_{1,t})\nu_j dSdt \\
=&\,\lambda\int_0^T\int_S \theta^2\sum_{i,j=1}^{n} a^{ij}(\mu\varphi\phi_{x_i}y_1y_{1,t} + \alpha_t y_1 y_{1,x_i})\nu_j dSdt \\
&+\lambda^2\mu\int_0^T\int_S \varphi\alpha_t\theta^2\sum_{i,j=1}^{n} a^{ij}\phi_{x_i}y_1^2\nu_j dSdt + \int_0^T\int_S \theta^2\sum_{i,j=1}^{n} a^{ij}y_{1,x_i}y_{1,t}\nu_j dSdt.
\end{aligned} \tag{4.8}$$

Next, we have

$$
\begin{aligned}
&-2\lambda\mu\int_0^T\int_{\Gamma_1}\varphi\sum_{i,j=1}^n a^{ij}v_{x_i}\sum_{k,\ell=1}^n a^{k\ell}\phi_{x_k}v_{x_\ell}\nu_j\ d\Gamma_1 dt\\
=&-2\lambda\mu\int_0^T\int_{\Gamma_1}\varphi\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}(\lambda\mu\varphi\phi_{x_i}y_1+y_{1,x_i})(\lambda\mu\varphi\phi_{x_\ell}y_1+y_{1,x_\ell})\nu_j d\Gamma_1 dt\\
=&-2\lambda^3\mu^3\int_0^T\int_S\varphi^3\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}\phi_{x_\ell}\phi_{x_k}y_1^2\nu_j dSdt\\
&-2\lambda\mu\int_0^T\int_{\Gamma_1}\varphi\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}y_{1,x_i}y_{1,x_\ell}\nu_j d\Gamma_1 dt\\
&-2\lambda^2\mu^2\int_0^T\int_S\varphi^2\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}y_1(\phi_{x_k}\phi_{x_\ell}y_{1,x_i}+\phi_{x_k}\phi_{x_i}y_{1,x_\ell})\nu_j dSdt.
\end{aligned}
\tag{4.9}
$$

Further,

$$
\begin{aligned}
&\lambda\mu\int_0^T\int_{\Gamma_1}\varphi\sum_{i,j=1}^n a^{ij}\phi_{x_i}\sum_{k,\ell=1}^n a^{k\ell}v_{x_k}v_{x_\ell}\nu_j d\Gamma_1 dt\\
=&\lambda\mu\int_0^T\int_{\Gamma_1}\varphi\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}(\lambda\mu\varphi\phi_{x_k}y_1+y_{1,x_k})(\lambda\mu\varphi\phi_{x_\ell}y_1+y_{1,x_\ell})\nu_j d\Gamma_1 dt\\
=&\lambda^3\mu^3\int_0^T\int_S\varphi^3\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}\phi_{x_k}\phi_{x_\ell}y_1^2\nu_j dSdt\\
&+2\lambda^2\mu^2\int_0^T\int_S\varphi^2\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}\phi_{x_\ell}y_{1,x_k}y_1\nu_j dSdt\\
&+\lambda\mu\int_0^T\int_{\Gamma_1}\varphi\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}y_{1,x_\ell}y_{1,x_k}\nu_j d\Gamma_1 dt.
\end{aligned}
\tag{4.10}
$$

Further,

$$
\begin{aligned}
&-2\lambda\mu^2\int_0^T\int_{\Gamma_1}\varphi\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}\phi_{x_\ell}v_{x_i}v\nu_j d\Gamma_1 dt\\
=&-2\lambda\mu^2\int_0^T\int_{\Gamma_1}\varphi\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}\phi_{x_\ell}(\lambda\mu\varphi\phi_{x_i}y_1+y_{1,x_i})y_1\nu_j d\Gamma_1 dt\\
=&-2\lambda^2\mu^3\int_0^T\int_S\varphi^2\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}\phi_{x_k}\phi_{x_\ell}y_1^2\nu_j dSdt\\
&-2\lambda\mu^2\int_0^T\int_S\varphi\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}\phi_{x_\ell}y_{1,x_i}y_1\nu_j dSdt.
\end{aligned}
\tag{4.11}
$$

Also, we have

$$
\begin{aligned}
&-\lambda^3\mu^3\int_0^T\int_{\Gamma_1}\varphi^3\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}\phi_{x_k}\phi_{x_\ell}v^2\nu_j d\Gamma_1 dt\\
&=-\lambda^3\mu^3\int_0^T\int_S\varphi^3\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}\phi_{x_\ell}\phi_{x_m}y_1^2\nu_j dSdt,
\end{aligned}
\tag{4.12}
$$

and

$$
\lambda^2\mu\int_0^T\int_{\Gamma_1}\varphi\alpha_t\sum_{i,j=1}^n a^{ij}\phi_{x_i}v^2\nu_j\ d\Gamma_1 dt=\lambda^2\mu\int_0^T\int_S\varphi\alpha_t\theta^2\sum_{i,j=1}^n a^{ij}\phi_{x_i}y_1^2\nu_j dSdt. \tag{4.13}
$$

From (4.7)–(4.13), we end up with

$$
\begin{aligned}
&\int_0^T\int_{\Omega_1}\operatorname{div}V dxdt\\
&=\int_0^T\int_S\theta^2\sum_{i,j=1}^n a^{ij}(\lambda\mu\varphi\phi_{x_i}y_1y_{1,t}+\lambda\alpha_t y_1 y_{1,x_i})\nu_j dSdt\\
&\quad+2\lambda^2\mu\int_0^T\int_S\varphi\alpha_t\theta^2\sum_{i,j=1}^n a^{ij}\phi_{x_i}y_1^2\nu_j dSdt+\int_0^T\int_S\theta^2\sum_{i,j=1}^n a^{ij}y_{1,x_i}y_{1,t}\nu_j dSdt\\
&\quad-2\lambda^2\mu^3\int_0^T\int_S(\lambda\varphi^3+\varphi^2)\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}\phi_{x_k}\phi_{x_\ell}y_1^2\nu_j dSdt\\
&\quad-\lambda\mu\int_0^T\int_{\Gamma_1}\varphi\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}[2\phi_{x_k}y_{1,x_i}-\phi_{x_i}y_{1,x_k}]y_{1,x_\ell}\nu_j d\Gamma_1 dt\\
&\quad-2\lambda\mu^2\int_0^T\int_S(\lambda\varphi^2+\varphi)\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}\phi_{x_\ell}y_{1,x_i}y_1\nu_j dSdt.
\end{aligned}
\tag{4.14}
$$

Step 2. Next, we set

$$
\hat\varphi=\frac{e^{-\mu\phi}}{t(T-t)},\quad \hat\alpha=\frac{e^{-\mu\phi}-e^{\mu(|\phi|_{L^\infty(\Omega_1)}+|\tilde\phi|_{L^\infty(\Omega_2)})}}{t(T-t)},\quad \hat\theta=e^{\lambda\hat\alpha}. \tag{4.15}
$$

It is easy to see that

$$
\varphi=\hat\varphi,\ \alpha=\hat\alpha,\ \theta=\hat\theta,\quad \text{on } (0,T)\times\Gamma_1. \tag{4.16}
$$

Using a similar argument to obtain (4.6) and (4.14), we can find three positive constants $\mu_3=\mu_3(\varepsilon)$, $\lambda_3=\lambda_3(\mu_3)$ and $C_3=C_3(\varepsilon)$, such that for all $\mu\ge\mu_3$ and $\lambda\ge\lambda_3$,

$$
\begin{aligned}
&C\int_0^T\int_{\Omega_1}\hat\theta^2 f_1^2 dxdt+C\int_0^T\int_{\omega_1}(\lambda\mu^2\hat\varphi|\nabla\hat v|^2+\lambda^3\mu^4\hat\varphi^3\hat v^2)dxdt\\
&\ge\int_0^T\int_{\Omega_1}\frac{\varepsilon}{8\lambda\hat\varphi}|\hat v_t|^2 dxdt+\int_0^T\int_{\Omega_1}\operatorname{div}\hat V dxdt\\
&\quad+C_3\int_0^T\int_{\Omega_1}(\lambda\mu^2\hat\varphi|\nabla\hat v|^2+\lambda^3\mu^4\hat\varphi^3\hat v^2)dxdt,
\end{aligned}
\tag{4.17}
$$

where $\hat{v}=\hat{\theta}y_1$, and

$$
\begin{aligned}
&\int_0^T\int_{\Omega_1}\operatorname{div}\hat{V}\,dxdt\\
=&\int_0^T\int_S\hat{\theta}^2\sum_{i,j=1}^n a^{ij}\big(-\lambda\mu\hat{\varphi}\phi_{x_i}y_1y_{1,t}+\lambda\hat{\alpha}_t y_1 y_{1,x_i}\big)\nu_j dSdt\\
&-2\lambda^2\mu\int_0^T\int_S\hat{\varphi}\hat{\alpha}_t\hat{\theta}^2\sum_{i,j=1}^n a^{ij}\phi_{x_i}y_1^2\nu_j dSdt+\int_0^T\int_S\hat{\theta}^2\sum_{i,j=1}^n a^{ij}y_{1,x_i}y_{1,t}\nu_j dSdt\\
&+2\lambda^2\mu^3\int_0^T\int_S(\lambda\hat{\varphi}^3+\hat{\varphi}^2)\hat{\theta}^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_i}\phi_{x_k}\phi_{x_\ell}y_1^2\nu_j dSdt\\
&+\lambda\mu\int_0^T\int_{\Gamma_1}\hat{\varphi}\hat{\theta}^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}[2\phi_{x_k}y_{1,x_i}-\phi_{x_i}y_{1,x_k}]y_{1,x_\ell}\nu_j d\Gamma_1 dt\\
&-2\lambda\mu^2\int_0^T\int_S(\lambda\hat{\varphi}^2+\hat{\varphi})\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}\phi_{x_\ell}y_{1,x_i}y_1\nu_j dSdt.
\end{aligned}
\tag{4.18}
$$

From (4.6), (4.14)and (4.17)–(4.18), and noting (4.16), we obtain that

$$
\begin{aligned}
&C\int_0^T\int_{\Omega_1}(\theta^2+\hat{\theta}^2)f_1^2dxdt+C\int_0^T\int_{\omega_1}(\lambda\mu^2\varphi|\nabla v|^2+\lambda^3\mu^4\varphi^3v^2)dxdt\\
&+C\int_0^T\int_{\omega_1}(\lambda\mu^2\hat{\varphi}|\nabla\hat{v}|^2+\lambda^3\mu^4\hat{\varphi}^3\hat{v}^2)dxdt\\
\ge&\int_0^T\int_{\Omega_1}\frac{\varepsilon}{8\lambda\varphi}|v_t|^2dxdt+\min(C_2,C_3)\int_0^T\int_{\Omega_1}(\lambda\mu^2\varphi|\nabla v|^2+\lambda^3\mu^4\varphi^3v^2)dxdt\\
&+2\lambda\int_0^T\int_S\theta^2\alpha_t\sum_{i,j=1}^n a^{ij}y_1y_{1,x_i}\nu_j dSdt+2\int_0^T\int_S\theta^2\sum_{i,j=1}^n a^{ij}y_{1,x_i}y_{1,t}\nu_j dSdt\\
&-4\lambda\mu^2\int_0^T\int_S(\lambda\varphi^2+\varphi)\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}\phi_{x_\ell}y_{1,x_i}y_1\nu_j dSdt.
\end{aligned}
\tag{4.19}
$$

It is clear that

$$
\frac{1}{C}\theta^2(|\nabla y_1|^2+\lambda\mu^2\varphi^2|y_1|^2)\le|\nabla v|^2+\lambda\mu^2\varphi^2v^2\le C\theta^2(|\nabla y_1|^2+\lambda\mu^2\varphi^2|y_1|^2)
\tag{4.20}
$$

and

$$
\frac{1}{C}\hat{\theta}^2(|\nabla y_1|^2+\lambda\mu^2\hat{\varphi}^2|y_1|^2)\le|\nabla\hat{v}|^2+\lambda\mu^2\hat{\varphi}^2\hat{v}^2\le C\hat{\theta}^2(|\nabla y_1|^2+\lambda\mu^2\hat{\varphi}^2|y_1|^2).
\tag{4.21}
$$

Choose an open non-empty subset ω_3 (of Ω_1) such that $\overline{\omega_1}\subset\omega_3$ and $\overline{\omega_3}\subset\omega\cap\Omega_1$, and a cut-off function $g\in C_0^\infty(\omega_3)$ satisfying $g=1$ in ω_1 and $0\le g\le 1$ in ω_3. Multiplying the first

equation in (1.4) by $g\theta^2\varphi y_1$ and integrating it in $(0,T)\times\Omega_1$, using integration by parts, we get

$$\int_0^T\int_{\omega_1}\varphi\theta^2|\nabla y_1|^2dxdt\le C\lambda^2\mu^2\int_0^T\int_{\omega_3}\theta^2\varphi^2y_1^2dxdt+C\int_0^T\int_{\Omega_1}\theta^2f_1^2dxdt. \tag{4.22}$$

Noting that

$$\varphi\ge\hat{\varphi}\ \text{ and }\ \theta\ge\hat{\theta},\qquad\text{in }(0,T)\times\Omega_1,$$

from (4.19)–(4.22), we obtain that

$$\begin{aligned}&C\int_0^T\int_{\Omega_1}\theta^2f_1^2dxdt+C\lambda^3\mu^4\int_0^T\int_{\omega\cap\Omega_1}\theta^2\varphi^3y_1^2dxdt\\ \ge&\int_0^T\int_{\Omega_1}\frac{\varepsilon}{8\lambda\varphi}|v_t|^2dxdt+\min(C_2,C_3)\int_0^T\int_{\Omega_1}\theta^2(\lambda\mu^2\varphi|\nabla y_1|^2+\lambda^3\mu^4\varphi^3y_1^2)dxdt\\ &+2\lambda\int_0^T\int_S\theta^2\alpha_t\sum_{i,j=1}^n a^{ij}y_1y_{1,x_i}\nu_jdSdt+2\int_0^T\int_S\theta^2\sum_{i,j=1}^n a^{ij}y_{1,x_i}y_{1,t}\nu_jdSdt\\ &-4\lambda\mu^2\int_0^T\int_S(\lambda\varphi^2+\varphi)\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n a^{ij}a^{k\ell}\phi_{x_k}\phi_{x_\ell}y_{1,x_i}y_1\nu_jdSdt.\end{aligned}\tag{4.23}$$

Step 3. Let $\widehat{\Omega}=\Omega_2$, $u=y_2$, $(b^{ij})_{1\le i,j\le n}=(\tilde{a}^{ij})_{1\le i,j\le n}$, $\psi=\tilde{\phi}$ and $d=|\phi|_{L^\infty(\Omega_1)}+|\tilde{\phi}|_{L^\infty(\Omega_2)}$ in Lemma 2.1. By a similar argument to obtain (4.23), for any small $\varepsilon>0$, there exist a $\mu_4=\mu_4(\varepsilon)$, a $\lambda_4=\lambda_4(\mu_4)$ and a positive constant $C_4=C_4(\varepsilon)$ such that for all $\mu\ge\mu_4$ and $\lambda\ge\lambda_4$, it holds that

$$\begin{aligned}&C\int_0^T\int_{\Omega_2}\tilde{\theta}^2f_2^2dxdt+C\lambda^3\mu^4\int_0^T\int_{\omega\cap\Omega_2}\tilde{\theta}^2\tilde{\varphi}^3y_2^2dxdt\\ \ge&\int_0^T\int_{\Omega_2}\frac{\varepsilon}{8\lambda\tilde{\varphi}}|(\tilde{\theta}y_2)_t|^2dxdt+C_4\int_0^T\int_{\Omega_2}\tilde{\theta}^2(\lambda\mu^2\tilde{\varphi}|\nabla y_2|^2+\lambda^3\mu^4\tilde{\varphi}^3y_2^2)dxdt\\ &-2\lambda\int_0^T\int_S\tilde{\theta}^2\tilde{\alpha}_t\sum_{i,j=1}^n\tilde{a}^{ij}y_2y_{2,x_i}\nu_jdSdt-2\int_0^T\int_S\tilde{\theta}^2\sum_{i,j=1}^n\tilde{a}^{ij}y_{2,x_i}y_{2,t}\nu_jdSdt\\ &+4\lambda\mu^2\int_0^T\int_S(\lambda\tilde{\varphi}^2+\tilde{\varphi})\tilde{\theta}^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n\tilde{a}^{ij}\tilde{a}^{k\ell}\tilde{\phi}_{x_k}\tilde{\phi}_{x_\ell}y_{2,x_i}y_2\nu_jdSdt,\end{aligned}\tag{4.24}$$

where we have used the fact that $\nu(x)=-\tilde{\nu}(x)$ on S.

By (4.23)–(4.24), we find that for all $\mu \geq \max\{\mu_1, \mu_2, \mu_3, \mu_4\}$ and $\lambda \geq \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$,

$$
\begin{aligned}
& C\int_0^T\int_{\Omega_1}\theta^2 f_1^2 dxdt + C\lambda^3\mu^4\int_0^T\int_{\omega\cap\Omega_1}\theta^2\varphi^3 y_1^2 dxdt \\
& +C\int_0^T\int_{\Omega_2}\tilde{\theta}^2 f_2^2 dxdt + C\lambda^3\mu^4\int_0^T\int_{\omega\cap\Omega_2}\tilde{\theta}^2\tilde{\varphi}^3 y_2^2 dxdt \\
\geq & \int_0^T\int_{\Omega_1}\frac{\varepsilon}{8\lambda\varphi}|v_t|^2 dxdt + \min(C_2, C_3)\int_0^T\int_{\Omega_1}\theta^2(\lambda\mu^2\varphi|\nabla y_1|^2 + \lambda^3\mu^4\varphi^3 y_1^2)dxdt \\
& +\int_0^T\int_{\Omega_2}\frac{\varepsilon}{8\lambda\tilde{\varphi}}|(\tilde{\theta}y_2)_t|^2 dxdt + C_4\int_0^T\int_{\Omega_2}\tilde{\theta}^2(\lambda\mu^2\tilde{\varphi}|\nabla y_2|^2 + \lambda^3\mu^4\tilde{\varphi}^3 y_2^2)dxdt \\
& +2\lambda\int_0^T\int_S\theta^2\alpha_t\sum_{i,j=1}^n(a^{ij}y_1y_{1,x_i} - \tilde{a}^{ij}y_2y_{2,x_i})\nu_j dSdt \\
& +2\int_0^T\int_S\theta^2\sum_{i,j=1}^n(a^{ij}y_{1,x_i}y_{1,t} - \tilde{a}^{ij}y_{2,x_i}y_{2,t})\nu_j dSdt \\
& -4\lambda\mu^2\int_0^T\int_S(\lambda\varphi^2+\varphi)\theta^2\sum_{i,j=1}^n\sum_{k,\ell=1}^n(a^{ij}a^{k\ell}\phi_{x_k}\phi_{x_\ell}y_{1,x_i}y_1 - \tilde{a}^{ij}\tilde{a}^{k\ell}\tilde{\phi}_{x_k}\tilde{\phi}_{x_\ell}y_{2,x_i}y_2)\nu_j dSdt.
\end{aligned}
\tag{4.25}
$$

From the transmission condition on S and the property of the weight functions ϕ and $\tilde{\phi}$, for any $\sigma > 0$, we conclude that

$$
\begin{aligned}
& \lambda\int_0^T\int_S\theta^2\alpha_t\sum_{i,j=1}^n(a^{ij}y_1y_{1,x_i} - \tilde{a}^{ij}y_2y_{2,x_i})\nu_j dSdt \\
= & \lambda\int_0^T\int_S\theta^2\alpha_t\Big[(y_2+\beta_1)\Big(\sum_{i,j=1}^n\tilde{a}^{ij}y_{2,x_i}\nu_j + \beta_2\Big) - \sum_{i,j=1}^n y_2\tilde{a}^{ij}y_{2,x_i}\nu_j\Big]dSdt \\
= & \lambda\int_0^T\int_S\theta^2\alpha_t\Big(\beta_1\sum_{i,j=1}^n\tilde{a}^{ij}y_{2,x_i}\nu_j + y_2\beta_2 + \beta_1\beta_2\Big)dSdt \\
\geq & \lambda\int_0^T\int_S\theta^2\alpha_t\beta_1\sum_{i,j=1}^n\tilde{a}^{ij}y_{2,x_i}\nu_j dSdt - \sigma\lambda\mu^2\int_0^T\int_S\varphi\theta^2 y_2^2 dSdt \\
& -\lambda\int_0^T\int_S\theta^2\Big(\frac{1}{\sigma\mu^2\varphi}|\alpha_t|^2\beta_2^2 + |\alpha_t\beta_1\beta_2|\Big)dSdt.
\end{aligned}
\tag{4.26}
$$

Also,

$$
\begin{aligned}
&\int_0^T \int_S \theta^2 \sum_{i,j=1}^n (a^{ij} y_{1,x_i} y_{1,t} - \tilde{a}^{ij} y_{2,x_i} y_{2,t}) \nu_j dSdt \\
&= \int_0^T \int_S \theta^2 \Big[\Big(\sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} \nu_j + \beta_2 \Big) (y_2 + \beta_1)_t - \sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} y_{2,t} \nu_j \Big] dSdt \\
&= \int_0^T \int_S \theta^2 \Big[\sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} \nu_j \beta_{1,t} + \beta_2 y_{2,t} + \beta_2 \beta_{1,t} \Big] dSdt \\
&= \int_0^T \int_S \theta^2 \Big[\sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} \nu_j \beta_{1,t} - (\beta_{2,t} + 2\lambda \alpha_t \beta_2) y_2 + \beta_2 \beta_{1,t} \Big] dSdt \\
&\geq \int_0^T \int_S \theta^2 \sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} \nu_j \beta_{1,t} dSdt - \sigma \lambda \mu^2 \int_0^T \int_S \varphi \theta^2 y_2^2 dSdt \\
&\quad - \int_0^T \int_S \theta^2 \Big(\frac{1}{\sigma \lambda \mu^2 \varphi} |\beta_{2,t} + 2\lambda \alpha_t \beta_2|^2 + |\beta_2 \beta_{1,t}| \Big) dSdt
\end{aligned}
\tag{4.27}
$$

and

$$
\begin{aligned}
&\lambda \mu^2 \int_0^T \int_S (\lambda \varphi^2 + \varphi) \theta^2 \sum_{i,j=1}^n \sum_{k,\ell=1}^n [a^{ij} a^{k\ell} \phi_{x_k} \phi_{x_\ell} y_{1,x_i} y_1 - \tilde{a}^{ij} \tilde{a}^{k\ell} \tilde{\phi}_{x_k} \tilde{\phi}_{x_\ell} y_{2,x_i} y_2] \nu_j dSdt \\
&= \lambda \mu^2 \int_0^T \int_S (\lambda \varphi^2 + \varphi) \theta^2 \sum_{k,\ell=1}^n a^{k\ell} \phi_{x_k} \phi_{x_\ell} \Big[\Big(\sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} \nu_j + \beta_2 \Big) (y_2 + \beta_1) \\
&\quad - \sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} \nu_j y_2 \Big] dSdt \\
&= \lambda \mu^2 \int_0^T \int_S (\lambda \varphi^2 + \varphi) \theta^2 \sum_{k,\ell=1}^n a^{k\ell} \phi_{x_k} \phi_{x_\ell} \Big(\sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} \nu_j \beta_1 + \beta_2 y_2 + \beta_1 \beta_1 \Big) dSdt \\
&\leq \lambda \mu^2 \int_0^T \int_S (\lambda \varphi^2 + \varphi) \theta^2 \sum_{k,\ell=1}^n a^{k\ell} \phi_{x_k} \phi_{x_\ell} \sum_{i,j=1}^n \tilde{a}^{ij} y_{2,x_i} \nu_j \beta_1 dSdt + \sigma \lambda \mu^2 \int_0^T \int_S \varphi \theta^2 y_2^2 dSdt \\
&\quad + C \sigma^{-1} \lambda^2 \mu^2 \int_0^T \int_S \theta^2 (\lambda \varphi^3 \beta_2^2 + |\beta_2 \beta_1|) dSdt.
\end{aligned}
\tag{4.28}
$$

Now, for any $\sigma > 0$, from (4.25)–(4.28), we conclude that

$$
\begin{aligned}
& C\int_0^T\int_{\Omega_1}\theta^2 f_1^2dxdt + C\lambda^3\mu^4\int_0^T\int_{\omega\cap\Omega_1}\theta^2\varphi^3 y_1^2dxdt\\
& +C\int_0^T\int_{\Omega_2}\tilde\theta^2 f_2^2dxdt + C\lambda^3\mu^4\int_0^T\int_{\omega\cap\Omega_2}\tilde\theta^2\tilde\varphi^3 y_2^2dxdt\\
& +C\Big|\int_0^T\int_S\theta^2\Big[\lambda\alpha_t\beta_1+\beta_{1,t}-2\lambda\mu^2(\lambda\varphi^2+\varphi)\sum_{k,\ell=1}^n a^{k\ell}\phi_{x_k}\phi_{x_\ell}\beta_1\Big]\sum_{i,j=1}^n\tilde a^{ij}y_{2,x_i}\nu_j dSdt\Big|\\
& +\sigma\lambda\mu^2\int_0^T\int_S\theta^2\varphi y_2^2dSdt + C\lambda\int_0^T\int_S\theta^2\Big(\frac{1}{\sigma\mu^2\varphi}|\alpha_t|^2\beta_2^2+|\alpha_t\beta_1\beta_2|\Big)dSdt\\
& +C\int_0^T\int_S\theta^2\Big(\frac{1}{\sigma\lambda\mu^2\varphi}|\beta_{2,t}|^2+|\beta_2\beta_{1,t}|\Big)dSdt + C\sigma^{-1}\lambda^2\mu^2\int_0^T\int_S\theta^2(\lambda\varphi^3\beta_2^2+|\beta_2\beta_1|)dSdt\\
\geq & \int_0^T\int_{\Omega_1}\frac{1}{\lambda\varphi}\theta^2|y_{1,t}|^2dxdt + \int_0^T\int_{\Omega_1}\theta^2(\lambda\mu^2\varphi|\nabla y_1|^2+\lambda^3\mu^4\varphi^3y_1^2)dxdt\\
& +\int_0^T\int_{\Omega_2}\frac{1}{\lambda\tilde\varphi}\tilde\theta^2|y_{2,t}|^2dxdt + \int_0^T\int_{\Omega_2}\tilde\theta^2(\lambda\mu^2\tilde\varphi|\nabla y_2|^2+\lambda^3\mu^4\tilde\varphi^3y_2^2)dxdt.
\end{aligned}
\tag{4.29}
$$

Step 4. By the trace theorem, it is clear that

$$
\begin{aligned}
& \int_0^T\int_S\theta^2\varphi y_2^2dSdt = \int_0^T|\tilde\theta\sqrt{\tilde\varphi}y_2|^2_{L^2(S)}dt \leq \int_0^T|\tilde\theta\sqrt{\tilde\varphi}y_2|^2_{L^2(\Gamma_2)}dt\\
& \leq C\int_0^T|\tilde\theta\sqrt{\tilde\varphi}y_2|^2_{H^1(\Gamma_2)}dt \leq C\int_0^T\int_{\Omega_2}\tilde\theta^2(\tilde\varphi|\nabla y_2|^2+\lambda^2\mu^2\tilde\varphi^3y_2^2)dxdt.
\end{aligned}
\tag{4.30}
$$

On the other hand, since $\beta_1\in L^2(0,T;H^{\frac32}(S))\cap H^1(0,T;H^{\frac12}(S))$, by means of the reverse trace theorem, one can find a function $\Xi\in L^2(0,T;H^2(\Omega_2))\cap H^1(0,T;H^1(\Omega_2))$ such that

$$
\Xi|_S=\beta_1,\qquad \Xi|_{\Gamma_2\setminus S}=0. \tag{4.31}
$$

Clearly,

$$
\begin{aligned}
& \tilde\theta^2\Big[\lambda\tilde\alpha_t\Xi+\Xi_t-2\lambda\mu^2(\lambda\tilde\varphi^2+\tilde\varphi)\sum_{k,\ell=1}^n\tilde a^{k\ell}\tilde\phi_{x_k}\tilde\phi_{x_\ell}\Xi\Big]\Big[y_{2,t}+\sum_{i,j=1}^n(\tilde a^{ij}y_{2,x_i})_{x_j}\Big]\\
= & \tilde\theta^2\Big[\lambda\tilde\alpha_t\Xi+\Xi_t-2\lambda\mu^2(\lambda\tilde\varphi^2+\tilde\varphi)\sum_{k,\ell=1}^n\tilde a^{k\ell}\tilde\phi_{x_k}\tilde\phi_{x_\ell}\Xi\Big]y_{2,t}\\
& +\sum_{i,j=1}^n\Big\{\tilde\theta^2\Big[\lambda\tilde\alpha_t\Xi+\Xi_t-2\lambda\mu^2(\lambda\tilde\varphi^2+\tilde\varphi)\sum_{k,\ell=1}^n\tilde a^{k\ell}\tilde\phi_{x_k}\tilde\phi_{x_\ell}\Xi\Big]\tilde a^{ij}y_{2,x_i}\Big\}_{x_j}\\
& -\sum_{i,j=1}^n\Big\{\tilde\theta^2\Big[\lambda\tilde\alpha_t\Xi+\Xi_t-2\lambda\mu^2(\lambda\tilde\varphi^2+\tilde\varphi)\sum_{k,\ell=1}^n\tilde a^{k\ell}\tilde\phi_{x_k}\tilde\phi_{x_\ell}\Xi\Big]\Big\}_{x_j}\tilde a^{ij}y_{2,x_i}.
\end{aligned}
\tag{4.32}
$$

Hence, by (4.32) and (1.4), and noting (4.31), we obtain that

$$
\begin{aligned}
&\Big|\int_0^T\int_S \theta^2\Big[\lambda\alpha_t\beta_1+\beta_{1,t}-2\lambda\mu^2(\lambda\varphi^2+\varphi)\sum_{k,\ell=1}^n a^{k\ell}\phi_{x_k}\phi_{x_\ell}\beta_1\Big]\sum_{i,j=1}^n \tilde a^{ij}y_{2,x_i}\nu_j dSdt\Big|\\
\le&\Big|\int_0^T\int_{\Omega_2}\tilde\theta^2\Big[\lambda\tilde\alpha_t\Xi+\Xi_t-2\lambda\mu^2(\lambda\tilde\varphi^2+\tilde\varphi)\sum_{k,\ell=1}^n\tilde a^{k\ell}\tilde\phi_{x_k}\tilde\phi_{x_\ell}\Xi\Big]f_2dxdt\Big|\\
&+\Big|\int_0^T\int_{\Omega_2}\tilde\theta^2\Big[\lambda\tilde\alpha_t\Xi+\Xi_t-2\lambda\mu^2(\lambda\tilde\varphi^2+\tilde\varphi)\sum_{k,\ell=1}^n\tilde a^{k\ell}\tilde\phi_{x_k}\tilde\phi_{x_\ell}\Xi\Big]y_{2,t}dxdt\Big|\\
&+\Big|\int_0^T\int_{\Omega_2}\sum_{i,j=1}^n\Big\{\tilde\theta^2\Big[\lambda\tilde\alpha_t\Xi+\Xi_t-2\lambda\mu^2(\lambda\tilde\varphi^2+\tilde\varphi)\sum_{k,\ell=1}^n\tilde a^{k\ell}\tilde\phi_{x_k}\tilde\phi_{x_\ell}\Xi\Big]\Big\}_{x_j}\tilde a^{ij}y_{2,x_i}dxdt\Big|\\
\le& C\int_0^T\int_{\Omega_2}\tilde\theta^2f_2^2dxdt+\int_0^T\int_{\Omega_2}\frac{1}{2\lambda\tilde\varphi}\tilde\theta^2|y_{2,t}|^2dxdt+\frac{\lambda\mu^2}{2}\int_0^T\int_{\Omega_2}\tilde\theta^2\tilde\varphi|\nabla y_2|^2dxdt\\
&+C\lambda\int_0^T\Big[\Big|\tilde\theta\tilde\varphi^{\frac12}\beta_{1,t}\Big|^2_{H^{\frac12}(S)}+\lambda^2\Big|\tilde\theta\tilde\varphi^{\frac12}|\tilde\alpha_t|\beta_1\Big|^2_{H^{\frac32}(S)}+\lambda^4\mu^4\Big|\tilde\theta\tilde\varphi^{\frac52}\beta_1\Big|^2_{H^{\frac32}(S)}\Big]dt.
\end{aligned}
\tag{4.33}
$$

By (4.29), (4.30) and (4.33), the desired estimate (4.2) follows. This completes the proof of Theorem 4.1. □

5 The case of time-dependent diffusion coefficients

In this section, we consider the general case of time-dependent diffusion coefficients.

For any parameters $\lambda>0$ and $\mu>0$, let

$$
\begin{aligned}
&\varphi^\ell=\frac{e^{\mu\phi^\ell}}{(t-t_\ell)(t_{\ell+1}-t)},\quad \alpha^\ell=\frac{e^{\mu\phi^\ell}-e^{\mu(|\phi^\ell|_{L^\infty((t_\ell,t_{\ell+1})\times\Omega_1)}+|\tilde\phi|_{L^\infty(\Omega_2)})}}{(t-t_\ell)(t_{\ell+1}-t)},\quad \theta^\ell=e^{\lambda\alpha^\ell}\\
&\tilde\varphi^\ell=\frac{e^{\mu\tilde\phi}}{(t-t_\ell)(t_{\ell+1}-t)},\quad \tilde\alpha^\ell=\frac{e^{\mu\tilde\phi}-e^{\mu(|\phi^\ell|_{L^\infty((t_\ell,t_{\ell+1})\times\Omega_1)}+|\tilde\phi|_{L^\infty(\Omega_2)})}}{(t-t_\ell)(t_{\ell+1}-t)},\quad \tilde\theta^\ell=e^{\lambda\tilde\alpha^\ell},
\end{aligned}
$$

where t_ℓ, ϕ^ℓ ($\ell=0,1,\cdots,L-1$), L and $\tilde\phi$ are given in Lemma 3.1. We have the following global Carleman estimate for solutions to the equation (1.4).

Theorem 5.1 *Assume that* (1.1)–(1.3) *and* (1.6) *hold. Then, there exists a $\mu_5>0$ such that for any $\mu\ge\mu_5$, one can find a $\lambda_5=\lambda_5(\mu)>0$ so that for any $\lambda\ge\lambda_5$, and each y_i^0*

$(i=1,2)$, f_i and β_i satisfying (1.5), *the corresponding solution to* (1.4) *satisfies*

$$
\begin{aligned}
&C\int_{t_\ell}^{t_{\ell+1}}\int_{\Omega_1}|\theta^\ell|^2 f_1^2dxdt + C\lambda^3\mu^4\int_{t_\ell}^{t_{\ell+1}}\int_{\omega\cap\Omega_1}|\theta^\ell|^2|\varphi^\ell|^3y_1^2dxdt\\
&+C\int_{t_\ell}^{t_{\ell+1}}\int_{\Omega_2}|\tilde\theta^\ell|^2 f_2^2dxdt + C\lambda^3\mu^4\int_{t_\ell}^{t_{\ell+1}}\int_{\omega\cap\Omega_2}|\tilde\theta^\ell|^2|\tilde\varphi^\ell|^3y_2^2dxdt\\
&+C\lambda\int_{t_\ell}^{t_{\ell+1}}\left[\left|\tilde\theta^\ell\sqrt{\tilde\varphi^\ell}\beta_{1,t}\right|^2_{H^{\frac12}(S)}+\lambda^2\left|\tilde\theta^\ell\sqrt{\tilde\varphi^\ell}|\tilde\alpha_t^\ell|\beta_1\right|^2_{H^{\frac32}(S)}+\lambda^4\mu^4\left|\tilde\theta^\ell|\tilde\varphi^\ell|^{\frac52}\beta_1\right|^2_{H^{\frac32}(S)}\right]dt\\
&+C\int_{t_\ell}^{t_{\ell+1}}\int_S\frac{|\theta^\ell|^2|\alpha_t^\ell|^2}{\mu\varphi^\ell}\beta_2^2dSdt+\int_{t_\ell}^{t_{\ell+1}}\int_S\frac{|\theta^\ell|^2}{\lambda\mu\varphi^\ell}|\beta_{2,t}|^2dSdt+C\lambda^3\mu^3\int_{t_\ell}^{t_{\ell+1}}\int_S|\theta^\ell|^2|\varphi^\ell|^3\beta_2^2dSdt\\
\geq&\int_{t_\ell}^{t_{\ell+1}}\int_{\Omega_1}\frac{1}{\lambda\varphi^\ell}|\theta^\ell|^2|y_{1,t}|^2dxdt+\int_{t_\ell}^{t_{\ell+1}}\int_{\Omega_1}|\theta^\ell|^2(\lambda\mu^2\varphi^\ell|\nabla y_1|^2+\lambda^3\mu^4|\varphi^\ell|^3y_1^2)dxdt\\
&+\int_{t_\ell}^{t_{\ell+1}}\int_{\Omega_2}\frac{1}{\lambda\tilde\varphi^\ell}|\tilde\theta^\ell|^2|y_{2,t}|^2dxdt+\int_{t_\ell}^{t_{\ell+1}}\int_{\Omega_2}|\tilde\theta^\ell|^2(\lambda\mu^2\tilde\varphi^\ell|\nabla y_2|^2+\lambda^3\mu^4|\tilde\varphi^\ell|^3y_2^2)dxdt,
\end{aligned}
\tag{5.1}
$$

where $\ell=0,1,\cdots,L-1$.

Proof. Replacing the time interval $[0,T]$ and weight functions ϕ, φ, α, θ, $\tilde\varphi$, $\tilde\alpha$ and $\tilde\theta$ (used in the proof of Theorem 4.1) accordingly by $[t_\ell,t_{\ell+1}]$, ϕ^ℓ, φ^ℓ, α^ℓ, θ^ℓ, $\tilde\varphi^\ell$, $\tilde\alpha^\ell$ and $\tilde\theta^\ell$, we obtain a proof of Theorem 5.1 (Hence we omit the details). □

Remark 5.1 *The estimate* (5.1) *is global in space but it is only "semi"-global in time. Nevertheless, from* (5.1), *it is enough to establish the usual observability for the equation* (1.4) *(and hence to prove the null controllability of parabolic equations with discontinuous and anisotropic diffusion coefficients). On the other hand, summing up the inequalities in* (5.1) *with respect to ℓ from 0 to $L-1$, one can obtain easily a global (both in time and space) Carleman estimate for the equation* (1.4).

Finally, we shall give a Carleman estimate (local in space but global in time) for the equation (1.4). For this purpose, we recall that, by [4, Lemma 1.1, p.4], one can find a function $\tilde\phi\in C^2(\overline{\Omega_2})$ satisfying (3.1) and a function $\phi\in C^2(\overline{\Omega_1})$ such that

$$
\begin{cases}
\phi>0 & \text{in } \Omega_1,\\
\phi=0 & \text{on } \Gamma_1,\\
|\nabla\phi|>0 & \text{in } \overline{\Omega_1\setminus\omega_1}.
\end{cases}
\tag{5.2}
$$

Without loss of generality, we may assume that $\overline{\omega_i}\cap S=\emptyset$ $(i=1,2)$. Since S is C^2, we may extend both a^{ij} and $\tilde a^{ij}$ $(1\le i,j\le n)$ to be C^2 functions on $[0,T]\times O_{\varepsilon_0}(S)$ (Here ε_0 is small enough such that the corresponding conditions like (1.2)-(1.3) are satisfied on $[0,T]\times O_{\varepsilon_0}(S)$ and $\overline{\omega_i}\cap O_{\varepsilon_0}(S)=\emptyset$), and still denote by a^{ij} and $\tilde a^{ij}$ accordingly the extended functions. Likewise, we extend both ϕ and $\tilde\phi$ to C^2 functions on $O_{\varepsilon_0}(S)$.

Let

$$
r=r(t,x)\triangleq\Big(\sum_{i,j=1}^n\tilde a^{ij}\tilde\phi_{x_i}\tilde\phi_{x_j}\Big)^{1/2},\qquad \tilde r=\tilde r(t,x)\triangleq\Big(\sum_{i,j=1}^n a^{ij}\phi_{x_i}\phi_{x_j}\Big)^{1/2}.
\tag{5.3}
$$

For any parameters $\lambda > 0$ and $\mu > 0$, put

$$\begin{aligned}
&\varphi = \frac{e^{\mu\phi r}}{t(T-t)}, \quad \alpha = \frac{e^{\mu\phi r} - e^{\mu(|\phi r|_{L^\infty((0,T)\times O_{\varepsilon_0}(S))} + |\tilde{\phi}\tilde{r}|_{L^\infty((0,T)\times O_{\varepsilon_0}(S))})}}{t(T-t)}, \quad \theta = e^{\lambda\alpha}, \\
&\tilde{\varphi} = \frac{e^{\mu\tilde{\phi}\tilde{r}}}{t(T-t)}, \quad \tilde{\alpha} = \frac{e^{\mu\tilde{\phi}\tilde{r}} - e^{\mu(|\phi r|_{L^\infty((0,T)\times O_{\varepsilon_0}(S))} + |\tilde{\phi}\tilde{r}|_{L^\infty((0,T)\times O_{\varepsilon_0}(S))})}}{t(T-t)}, \quad \tilde{\theta} = e^{\lambda\tilde{\alpha}}.
\end{aligned} \tag{5.4}$$

We have the following local (in space) Carleman estimate for solutions to the equation (1.4) (Here we consider only the case of Carleman estimate which is locally in a neighborhood of the interface S).

Theorem 5.2 *Assume that* (1.1)–(1.3) *and* (1.6) *hold. Then, there exist a neighborhood $O(S)(\subset O_{\varepsilon_0}(S))$ of S and a $\mu_6 > 0$ such that for any $\mu \geq \mu_6$, one can find a $\lambda_6 = \lambda_6(\mu) > 0$ so that for any $\lambda \geq \lambda_6$, and each y_i^0 $(i = 1, 2)$, f_i and β_i satisfying* (1.5), *if the corresponding solution to* (1.4) *is supported (in space) in $O(S)$, then*

$$\begin{aligned}
&C\int_0^T\int_{\Omega_1\cap O(S)} \theta^2 f_1^2 dxdt + C\int_0^T\int_{\Omega_2\cap O(S)} \tilde{\theta}^2 f_2^2 dxdt \\
&+C\lambda\int_0^T \left[\left|\tilde{\theta}\tilde{\varphi}^{\frac{1}{2}}\beta_{1,t}\right|^2_{H^{\frac{1}{2}}(S)} + \lambda^2\left|\tilde{\theta}\tilde{\varphi}^{\frac{1}{2}}|\tilde{\alpha}_t|\beta_1\right|^2_{H^{\frac{3}{2}}(S)} + \lambda^4\mu^4\left|\tilde{\theta}\tilde{\varphi}^{\frac{5}{2}}\beta_1\right|^2_{H^{\frac{3}{2}}(S)} \right] dt \\
&+C\int_0^T\int_S \frac{\theta^2|\alpha_t|^2}{\mu\varphi}\beta_2^2 dSdt + \int_0^T\int_S \frac{\theta^2}{\lambda\mu\varphi}|\beta_{2,t}|^2 dSdt + C\lambda^3\mu^3\int_0^T\int_S \theta^2\varphi^3\beta_2^2 dSdt \\
\geq &\int_0^T\int_{\Omega_1\cap O(S)} \frac{1}{\lambda\varphi}\theta^2|y_{1,t}|^2 dxdt + \int_0^T\int_{\Omega_1\cap O(S)} \theta^2(\lambda\mu^2\varphi|\nabla y_1|^2 + \lambda^3\mu^4\varphi^3 y_1^2) dxdt \\
&+\int_0^T\int_{\Omega_2\cap O(S)} \frac{1}{\lambda\tilde{\varphi}}\tilde{\theta}^2|y_{2,t}|^2 dxdt + \int_0^T\int_{\Omega_2\cap O(S)} \tilde{\theta}^2(\lambda\mu^2\tilde{\varphi}|\nabla y_2|^2 + \lambda^3\mu^4\tilde{\varphi}^3 y_2^2) dxdt.
\end{aligned} \tag{5.5}$$

Proof. It is clear that $\phi r = \tilde{\phi}\tilde{r} = 0$ on $[0,T]\times S$. Also, by (5.3) and the fact that $\phi = \tilde{\phi} = 0$ on S, it follows that

$$\sum_{i,j=1}^n a^{ij}(\phi r)_{x_i}(\phi r)_{x_j} = \sum_{i,j=1}^n \tilde{a}^{ij}(\tilde{\phi}\tilde{r})_{x_i}(\tilde{\phi}\tilde{r})_{x_j}, \quad \text{ on } [0,T]\times S.$$

By (3.1), (5.2) and (5.3), it is easy to see that $\frac{\partial(\phi r)}{\partial\nu} < 0$ and $\frac{\partial(\tilde{\phi}\tilde{r})}{\partial\nu} > 0$ on $[0,T]\times S$. Hence, one can find an $\varepsilon_1 \in (0, \varepsilon_0)$ and a constant $c_0 > 0$ such that

$$|\nabla(\phi r)| > c_0, \quad |\nabla(\tilde{\phi}\tilde{r})| > c_0 \qquad \text{in } [0,T]\times O_{\varepsilon_1}(S).$$

We choose $O(S) = O_{\varepsilon_1}(S)$.

Now, replacing the space domains Ω_i $(i = 1, 2)$ and the weight functions ϕ and $\tilde{\phi}$ (used in the proof of Theorem 4.1) accordingly by $\Omega_i \cap O(S)$, ϕr and $\tilde{\phi}\tilde{r}$, we obtain a proof of Theorem 5.2 (Hence we omit the details). $\square$

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